MINISTRY OF EDUCATION MALAYSIA

DUAL LANGUAGE PROGRAMME

MATHEMATICS

Form 5

KURIKULUM STANDARD SEKOLAH MENENGAH
KSSM
KEMENTERIAN PENDIDIKAN MALAYSIA

RUKUN NEGARA

Bahawasanya Negara Kita Malaysia
mendukung cita-cita hendak;

Mencapai perpaduan yang lebih erat dalam kalangan
seluruh masyarakatnya;

Memelihara satu cara hidup demokrasi;

Mencipta satu masyarakat yang adil di mana kemakmuran negara
akan dapat dinikmati bersama secara adil dan saksama;

Menjamin satu cara yang liberal terhadap
tradisi-tradisi kebudayaannya yang kaya dan pelbagai corak;

Membina satu masyarakat progresif yang akan menggunakan
sains dan teknologi moden;

MAKA KAMI, rakyat Malaysia,
berikrar akan menumpukan
seluruh tenaga dan usaha kami untuk mencapai cita-cita tersebut
berdasarkan prinsip-prinsip yang berikut:

KEPERCAYAAN KEPADA TUHAN
KESETIAAN KEPADA RAJA DAN NEGARA
KELUHURAN PERLEMBAGAAN
KEDAULATAN UNDANG-UNDANG
KESOPANAN DAN KESUSILAAN

(Sumber: Jabatan Penerangan, Kementerian Komunikasi dan Multimedia Malaysia)

KURIKULUM STANDARD SEKOLAH MENENGAH
DUAL LANGUAGE PROGRAMME

MATHEMATICS

FORM 5

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Penerbitan Pelangi Sdn. Bhd.
2020

MINISTRY OF EDUCATION MALAYSIA

BOOK SERIES NO: 0110

KPM2020 ISBN 978-967-2930-12-9
First Published 2020

Published for Ministry of Education Malaysia by:
PENERBITAN PELANGI SDN. BHD.
66, Jalan Pingai, Taman Pelangi,
80400 Johor Bahru,
Johor Darul Takzim
Tel: 07-3316288
E-mail: pelangi@pelangibooks.com
Website: www.pelangibooks.com

Layout and Typesetting:
PENERBITAN PELANGI SDN. BHD.
Font Type: Times New Roman
Font Size:11 points

Printed by:
Comtech Marketing Sdn. Bhd.
16, Jalan Bukit 2,
Kawasan MIEL,
Bandar Seri Alam,
81750 Masai, Johor Darul Takzim.

ACKNOWLEDGEMENT

The publishing of this textbook involves cooperation from various parties. Our wholehearted appreciation and gratitude go out to all the parties involved:

- Committee members of *Penambahbaikan Pruf Muka Surat*, Educational Resources and Technology Division, Ministry of Education Malaysia.
- Committee members of *Penyemakan Naskhah Sedia Kamera*, Educational Resources and Technology Division, Ministry of Education Malaysia.
- Committee members of *Penyemakan Naskhah Sedia Kamera* for Dual Language Programme, Educational Resources and Technology Division, Ministry of Education Malaysia.
- Committee members of *Penyemakan Pembetulan Naskhah Sedia Kamera* for Dual Language Programme, Educational Resources and Technology Division, Ministry of Education Malaysia.
- Officers of the Educational Resources and Technology Division, the Curriculum Development Division, and the English Language Teaching Centre, Ministry of Education Malaysia.
- Chairperson and members of the Quality Control Panel.
- Editorial Team and Production Team, especially the illustrator and designer.
- International GeoGebra Institute
- Freepik.com
- Everyone who has been directly or indirectly involved in the publication of this book.

Contents

Introduction

This **Form 5 Mathematics KSSM Textbook** is written based on *Dokumen Standard Kurikulum dan Pentaksiran (DSKP)*. This book meets the aims of *Kurikulum Standard Sekolah Menengah (KSSM)* in applying mathematical knowledge and skills so that the pupils are able to deal with challenges in their daily lives, in line with the development of science and technology as well as the challenges of the 21st century.

The structure of this book emphasises on HOTS, STEM, EMK and the use of digital technology focusing on Inquiry-based Learning and Project-based Learning approaches to stimulate pupils' mind and intellect to compete globally.

This book has 8 chapters to be explored and each chapter consists of Stimulus Page, Content and Ending which highlight special features.

STIMULUS PAGE

What will you learn?
Consists of the Content Standards of the chapter.

Why study this chapter?
Exposes pupils to the real-life and career related applications.

Do you know?
Displays the development of mathematics as well as the contribution of great mathematicians as an added knowledge to pupils.

Word Bank
List of keywords to learn together with its translation in Bahasa Malaysia.

CONTENT

MIND MOBILISATION
Explains the learning concepts through the active involvement of pupils and encourages mathematical communication through discussion.

PROJECT
Enables pupils to apply the knowledge they have learnt to the problem in reality and present their outcomes during the lesson.

Self Practice
Tests the understanding of the pupils on the concepts they have learnt.

Indicates HOTS questions to test pupils' higher order thinking skills.

MEMORY BOX

Helps pupils remember the formulae or concepts they have learnt.

Info Bulletin

Consists of mathematical information as extra knowledge that related to the chapter.

Smart TiPS

Provides effective and helpful tips or guides.

Application & Career

Shows the application of concepts learnt in real life and careers.

Historical Treasure

Explains the history related to the title as extra knowledge.

Critical Mind

Poses questions that stimulate pupils to think outside the box.

i-Technology

Guides pupils in the use of technology in learning.

Interactive Platform

Encourages discussion between teachers with pupils and pupils with pupils.

Mathematics is fun!

Encourages fun mathematics learning.

Checking Answer

Guides pupils to check the answers.

Alternative Method

Shows solutions using other methods.

My Malaysia

Shows the knowledge related to the title and the development of Malaysia.

ENDING

Summary Arena
Consists of relations between concepts in mind maps.

Extensive Practice
A summative assessment which consists of Understand, Mastery and Challenge.

Exploring Mathematics
Fun and relaxing concept exploration activities.

Reflection
Helps pupils gauge the level of mastery of the topics in the chapter.

Mini Project
Enhances pupils' overall understanding.

Symbols and Formulae

Symbols

$=$	is equal to
$\neq$	is not equal to
$\approx$	is approximately equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$<$	is less than
$\leqslant$	is less than or equal to
Δ	triangle
$\angle$	angle
$\propto$	varies directly as
sin	sine
cos	cosine
tan	tangent
$^\circ$	degree
$'$	minute (angle)
Σ	sum
$\bar{x}$	mean
σ^2	variance
σ	standard deviation

Formulae

$$\text{Premium} = \frac{\text{Face value of policy}}{\text{RM}x} \times \begin{pmatrix}\text{Premium rate}\\ \text{per RM}x\end{pmatrix}$$

Amount of required insurance

$$= \begin{pmatrix}\text{Percentage of}\\ \text{co-insurance}\end{pmatrix} \times \begin{pmatrix}\text{Insurable value}\\ \text{of property}\end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Scale factor, } k = \frac{PA'}{PA}$$

$$\text{Area of image} = k^2 \times \text{Area of object}$$

$$\text{Mean, } \bar{x} = \frac{\sum fx}{\sum f}$$

$$\text{Variance, } \sigma^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2$$

$$\text{Standard deviation, } \sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

Guide to Access Digital Materials in This Book

Download the free QR code scanning application to your smartphone. Scan the QR code or visit *http://bukutekssm.my/Mathematics/F5/Index.html* to access the digital materials such as:

- Worksheets
- GeoGebra files
- Videos

Then download the materials for offline use.

Note: Pupils can download the free GeoGebra software to open the relevant file via *http://www.geogebra.org/*

CHAPTER 1 Variation

What will you learn?

- Direct Variation
- Inverse Variation
- Combined Variation

Why study this chapter?

The concept of variation is commonly used in our daily life and in the fields of science and mathematics. For example, finding the distance travelled in terms of speed and given time, investigating the relation between the depth of the sea and the water pressure and others.

Do you know?

The symbol $\propto$ was introduced by William Emerson (1701-1782), an English mathematician, in 1768 in his book, The Doctrine of Fluxions.

For more information:

bit.do/DoYouKnowChap1

WORD BANK

constant	*pemalar*
variable	*pemboleh ubah*
combined variation	*ubahan bergabung*
direct variation	*ubahan langsung*
inverse variation	*ubahan songsang*
joint variation	*ubahan tercantum*

In scientific research, electricians conduct experiments to investigate the relation between two or more variables. For example, in an experiment on electric current, if the value of voltage increases, then the value of the current flowing through the circuit increases too and vice versa. In contrast, if the value of the resistance decreases, then the value of the current increases and vice versa.
Do you know how the current (I), voltage (V) and resistance (R) relate to each other?

1.1 Direct Variation

What is the meaning of direct variation?

Learning Standard

Explain the meaning of direct variation.

5 cups of rice for 10 servings. 10 cups of rice for 20 servings.

A chef needs to decide the amount of ingredients to be used according to the number of servings. If the ingredients used increase, then the number of servings increases. On the contrary, if the ingredients used decrease, then the number of servings decreases too.

In our daily life, we often face situations that involve the relation of quantity changes. For example, the distance travelled and the taxi fare, the amount of interest earned over a period of time at a certain interest rate.

MIND MOBILISATION 1 Individual

Aim: To explain the meaning of direct variation.

Steps:

1. By browsing through the Bank Negara Malaysia website, find the latest rate of currency exchange of Malaysian Ringgit (RM) to Singapore Dollar ($), Thailand Baht (฿) and Japanese Yen (¥).
2. By using the rate of currency exchange, calculate the following values of Malaysian Ringgit to Singapore Dollar, Thailand Baht and Japanese Yen.

Malaysian Ringgit (RM)	10	20	30	40	50
Singapore Dollar ($)					
Thailand Baht (฿)					
Japanese Yen (¥)					

Discussion:

1. State the change
   (a) to Singapore Dollar, Thailand Baht and Japanese Yen when Malaysian Ringgit increases,
   (b) to Singapore Dollar, Thailand Baht and Japanese Yen when Malaysian Ringgit divides by four,
   (c) to Singapore Dollar, Thailand Baht and Japanese Yen when Malaysian Ringgit is doubled,
   (d) to Singapore Dollar, Thailand Baht and Japanese Yen when Malaysian Ringgit decreases by 50%.
2. What is the relation between the value of Malaysian Ringgit and the values of Singapore Dollar, Thailand Baht and Japanese Yen?

The results of Mind Mobilisation 1 show that the change in the value of Malaysian Ringgit, RM corresponds to the change in the values of Singapore Dollar, $, Thailand Baht, ฿ and Japanese Yen, ¥. The value of RM increases when the values of $, ฿ and ¥ increase, and the value of RM decreases when the values of $, ฿ and ¥ decrease.

This relation is known as a direct variation. In general,

Direct variation explains the relation between two variables, such that when variable y increases, then variable x also increases at the same rate and vice versa. This relation can be written as y varies directly as x.

Example 1

The total wage of a part time worker as a promoter varies directly as the number of hours he works. State the change on
(a) the total wage if the number of working hours is doubled,
(b) the total wage if the number of working hours decreases by 40%,
(c) the number of working hours if the total wage is halved.

Solution:

(a) The total wage is doubled.
(b) The total wage decreases by 40%.
(c) The number of working hours is halved.

Self Practice 1.1a

1. The value of resistance of a wire varies directly as its temperature. State the change on
   (a) the value of resistance if the temperature increases by 10%,
   (b) the value of resistance if the temperature decreases by half,
   (c) the temperature if the value of the resistance decreases by $\frac{1}{4}$.

2. Puan Wardina wants to buy green beans selling at RMx per kilogram. State the price if Puan Wardina buys
   (a) 500 g of green beans,
   (b) 2 kg of green beans.

3. The table below shows the relation between the time needed and the number of bottles of jam produced in a factory.

Time (minutes)	5	10	15	20	25
Number of bottles	10	20	30	40	50

State the change in the number of bottles of jam produced when
(a) the time is doubled,
(b) the time is halved.

What is the relation between two variables for a direct variation?

Learning Standard

Determine the relation between two variables for a direct variation.

MIND MOBILISATION 2 Group

Aim: To determine the relation between two variables for a direct variation.

Steps:

1. Divide the class into six groups.
2. Each group chooses a type of rate from the list below.
   (a) Rate of taxi fare by distance
   (b) Rate of parking by hours
   (c) Rate of heartbeat by minutes
   (d) Rate of toll by distance
   (e) Rate of data downloading (Mbps)
   (f) Rate of simple interest in savings per year
3. By browsing the internet, get information on the chosen rate and complete the table as shown below.
   For example, the rate of taxi fare is RM1.20 per km.

Distance travelled, x (km)	2	4	6	8	10
Fare, y (RM)					
$\frac{y}{x}$					

4. Draw the graph of y against x using a suitable scale based on the table.
5. Answer the questions in Discussion.
6. Present the results to the class.

Discussion:

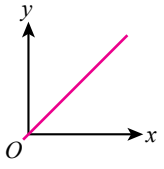
1. Based on the values of $\frac{y}{x}$, what conclusion can be made?
2. State the shape of the graph obtained.
3. What is the relation between variable y and variable x?

The results of Mind Mobilisation 2 show that for a direct variation,

(a) the value of $\frac{y}{x}$ is a constant,

(b) the graph of y against x is a straight line that passes through the origin,

(c) the variable y varies directly as the variable x.



In direct variation, quantity y is said to be varied directly as x if and only if $\frac{y}{x}$ is a constant, known as the constant of proportionality, k.

Relation between the value of k and the concept of proportion:

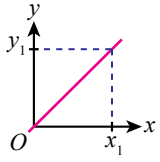
y	y_1	y_2	y_3	y_4
x	x_1	x_2	x_3	x_4

$$k = \frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \frac{y_4}{x_4}$$

Relation between the value of k and the gradient of a straight line passes through the origin:

Gradient,

$$m = \frac{y_1}{x_1}$$
$$= k$$



This relation is true only if the straight line passes through the origin.

When y varies directly as x, then the relation is written as $y \propto x$. From this relation, the value of constant of proportionality k can be determined, that is

$$k = \frac{y}{x}.$$

For a **direct variation**, y varies directly as x can be written as

$$y \propto x \quad \text{(variation relation)}$$
$$y = kx \quad \text{(equation form)}$$

where k is a constant.

Smart TiPS

The graph of y against x is a graph with variable y representing the vertical axis and variable x representing the horizontal axis.

Info Bulletin

Constant is a fixed or unchanged quantity value.

Critical Mind

Does every gradient value represent the value of proportionality, k? Discuss.

Smart TiPS

The relation of direct variation written as 'y varies directly as x' can also be written as 'x and y are directly proportional'.

Smart TiPS

The symbol $\propto$ means 'is proportional to'.

MIND MOBILISATION 3 Group

Aim: To determine the relation between the variables y and x^2 for a direct variation.

Steps:

1. Given a circle with radius, x cm, and area, y cm².
2. Take turns to complete the table below in your group. (Use $\pi = \frac{22}{7}$)

Radius, x (cm)	3.5	7.0	10.5	14.0	17.5
Area of circle, y (cm²)					
Square of radius, x^2 (cm²)					
Cube of radius, x^3 (cm³)					
$\frac{y}{x^2}$					
$\frac{y}{x^3}$					

3. Draw the graph of y against x^2 and the graph of y against x^3 by using approximate values.

Discussion:

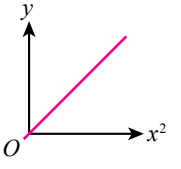
1. Compare the values of $\frac{y}{x^2}$ and $\frac{y}{x^3}$. What conclusion can be made?
2. Which graph is a straight line graph that passes through the origin?
3. What is the relation between variable y and variable x^2?

The results of Mind Mobilisation 3 show that for a direct variation,

(a) the value of $\frac{y}{x^2}$ is a constant. Hence, $k = \frac{y}{x^2}$,

(b) the graph of y against x^2 is a straight line that passes through the origin,

(c) variable y varies directly as variable x^2.



In general,

For a **direct variation**, y varies directly as x^n can be written as

$y \propto x^n$ (variation relation)

$y = kx^n$ (equation form)

where $n = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$ and k is a constant.

The graph of y against x^n is a straight line passes through the origin where k is the gradient of the straight line.

Critical Mind

In Mind Mobilisation 3, what is the relation between the values of π and $\frac{y}{x^2}$?

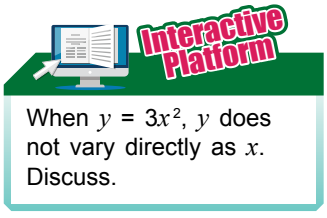
Smart TIPS

$y \propto x$ is true if and only if $n = 1$.

Example 2

A toy car moves from a state of rest. The distance travelled by the toy car, y, changes with time, t, as shown in the table below.

Time, t (s)	2	4	6	8	10	12
Distance, y (cm)	14	28	42	56	70	84



Determine whether y varies directly as t or t^2. Hence, write the relation in the form of variation.

Solution:

t	2	4	6	8	10	12
y	14	28	42	56	70	84
$\frac{y}{t}$	$\frac{14}{2} = 7$	$\frac{28}{4} = 7$	$\frac{42}{6} = 7$	$\frac{56}{8} = 7$	$\frac{70}{10} = 7$	$\frac{84}{12} = 7$
$\frac{y}{t^2}$	$\frac{14}{2^2} = 3.50$	$\frac{28}{4^2} = 1.75$	$\frac{42}{6^2} = 1.17$	$\frac{56}{8^2} = 0.88$	$\frac{70}{10^2} = 0.70$	$\frac{84}{12^2} = 0.58$

y varies directly as t because the value of $\frac{y}{t}$ is a constant. Hence, $y \propto t$.

y does not vary directly as t^2 because $\frac{y}{t^2}$ is not a constant.

Example 3

(a) By drawing the graph of y against x, determine whether y varies directly as x.

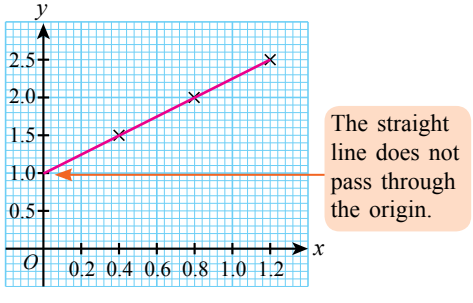
x	0.4	0.8	1.2
y	1.5	2.0	2.5

(b) By drawing the graph of y against x^2, determine whether y varies directly as x^2.

x	2	3	4	5
y	8	18	32	50

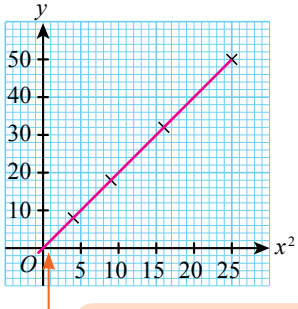
Solution:

(a)



Hence, y does not vary directly as x.

(b)



x^2	y
4	8
9	18
16	32
25	50

The straight line passes through the origin.

Hence, y varies directly as x^2.

Example 4

It is given $m = 0.8$ when $n = 0.125$. Express m in terms of n if
(a) m varies directly as n,
(b) m varies directly as the cube root of n.

Solution:

(a) $m \propto n$

$m = kn$ ← Write the relation in the form of equation.

$0.8 = k(0.125)$ ← Substitute the values of m and n into the equation to obtain the value of k.

$k = \dfrac{0.8}{0.125}$

$= 6.4$

Hence, $m = 6.4n$

(b) $m \propto \sqrt[3]{n}$

$m = k\sqrt[3]{n}$

$0.8 = k(\sqrt[3]{0.125})$

$k = \dfrac{0.8}{\sqrt[3]{0.125}}$

$= 1.6$

Hence, $m = 1.6\sqrt[3]{n}$

Example 5

The extension of a spring, x cm, varies directly as the mass of the load, w g, hanging on the string. It is given the extension of a spring is 3 cm when a load of 200 g is hung on the string. Express x in terms of w.

Solution:

$x \propto w$

$x = kw$ ← Write the relation in the form of equation.

$3 = k(200)$ ← Substitute the values of x and w into the equation to obtain the value of k.

$k = \dfrac{3}{200}$

$= 0.015$

Hence, $x = 0.015w$

Example 6

It is given y varies directly as x. If $y = 0.14$ when $x = 0.2$, calculate the value of
(a) y when $x = 5$,
(b) x when $y = 0.875$.

Solution:

$y \propto x$

$y = kx$

$0.14 = k(0.2)$

$k = \dfrac{0.14}{0.2}$

$= 0.7$

Hence, $y = 0.7x$

(a) When $x = 5$,

$y = 0.7(5)$

$= 3.5$

(b) When $y = 0.875$,

$0.875 = 0.7x$

$x = \dfrac{0.875}{0.7}$

$= 1.25$

Critical Mind

In the graph of Example 3(a), why does y not vary directly as x?

Smart TIPS

(i) y varies directly as x, $y \propto x$

(ii) y varies directly as the square of x, $y \propto x^2$

(iii) y varies directly as the cube of x, $y \propto x^3$

(iv) y varies directly as the square root of x, $y \propto \sqrt{x}$

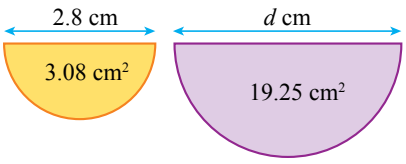
(v) y varies directly as the cube root of x, $y \propto \sqrt[3]{x}$

Historical Treasure

Robert Hooke (1635-1703) is a British scientist who introduced Hooke's Law in 1676. The law states that the extension of an elastic object is directly proportional to the force applied to stretch it, as long as it does not exceed the elastic limit.

Example 7

The area, L cm^2, of a semicircle varies directly as the square of its diameter, d cm. It is given that the area of the semicircle is 3.08 cm^2 when the diameter is 2.8 cm. Calculate the value of d when $L = 19.25$.



Solution:

$$L \propto d^2$$
$$L = kd^2$$
$$3.08 = k(2.8)^2$$
$$k = \frac{3.08}{(2.8)^2}$$
$$= \frac{11}{28}$$

Hence, $L = \frac{11}{28}d^2$

When $L = 19.25$,

$$19.25 = \frac{11}{28}d^2$$
$$d^2 = \frac{19.25 \times 28}{11}$$
$$d = \sqrt{49}$$
$$= 7 \text{ cm}$$

Alternative Method:

Using the concept of proportion:
Given that $L_1 = 3.08$, $d_1 = 2.8$ and $L_2 = 19.25$

$$\frac{L_1}{(d_1)^2} = \frac{L_2}{(d_2)^2}$$
$$\frac{3.08}{2.8^2} = \frac{19.25}{(d_2)^2}$$
$$(d_2)^2 = \frac{19.25 \times 2.8^2}{3.08}$$
$$d_2 = \sqrt{49}$$
$$= 7 \text{ cm}$$

Example 8

The period of oscillation, A seconds, of a simple pendulum varies directly as the square root of the length of its string, p cm. It is given that the 9 cm string of a simple pendulum oscillates for 1.2 seconds. Calculate the period of oscillation in seconds, if the length of the string is 25 cm.

Solution:

$$A \propto \sqrt{p}$$
$$A = k\sqrt{p}$$
$$1.2 = k\sqrt{9}$$
$$k = \frac{1.2}{\sqrt{9}}$$
$$= 0.4$$

Hence, $A = 0.4\sqrt{p}$

When $p = 25$,

$$A = 0.4\sqrt{25}$$
$$= 2 \text{ seconds}$$

Alternative Method:

Using the concept of proportion:
Given that $A_1 = 1.2$, $p_1 = 9$ and $p_2 = 25$

$$\frac{A_1}{\sqrt{p_1}} = \frac{A_2}{\sqrt{p_2}}$$
$$\frac{1.2}{\sqrt{9}} = \frac{A_2}{\sqrt{25}}$$
$$A_2 = \frac{1.2 \times \sqrt{25}}{\sqrt{9}}$$
$$= 2 \text{ seconds}$$

Self Practice 1.1b

1. The table below shows the values of two variables, x and y.

(a) Determine whether y varies directly as x or x^3. Then, write the relation in the form of variation.

x	1	2	3	4	5
y	2.5	5	7.5	10	12.5

(b) Determine whether y varies directly as x or $\sqrt{x}$. Then, write the relation in the form of variation.

x	4	9	25	36	49
y	0.6	0.9	1.5	1.8	2.1

2. A load is hanging on a spring. The table on the right shows the mass of the load, x g, with the extension of the spring, p cm. By drawing the graph of p against x, determine whether p varies directly as x.

Mass of load, x (g)	5	10	15	25	30
Extension of spring, p (cm)	1	2	3	5	6

3. It is given that $p = 32$ when $q = 4$. Express p in terms of q if
(a) p varies directly as q^3,
(b) p varies directly as the square root of q.

4. The salary, RMx, obtained by a worker varies directly as the total time he works, t hours. It is given a worker received RM112 after he worked for 14 hours. Write an equation that relates x and t.

5. Given that $y = 1.8$ when $x = 0.6$, calculate the value of y when $x = 5$ if
(a) $y \propto x$,
(b) $y \propto x^2$.

6. It is given that s varies directly as $t^{\frac{1}{3}}$. If $s = 1.2$ when $t = 27$, calculate the value of
(a) s when $t = 64$,
(b) t when $s = 0.28$.

7. The number of words typed, a, by Saiful varies directly as the time he types, t minutes. If Saiful types 270 words in 6 minutes, calculate the time he needs to spend to type 675 words.

8. An object falls from a height, h m, varies directly as the square of its time, t s on planet Q. Given that the object falls from the height of 5 m in 2 s, calculate the time taken in seconds, for the object to fall from a height of 45 m on the planet.

9. It is given that the volume of the paint, x litres, varies directly as the area of the wall, d m^2. If 3 litres of paint can paint 36 m^2 of wall,
(a) express the equation in terms of x and d,
(b) calculate the volume of the paint in litres, needed to paint a wall of 9 m in height and 5 m in width.

What is the relation between three or more variables for a given joint variation?

Learning Standard

Determine the relation between three or more variables for a given joint variation.

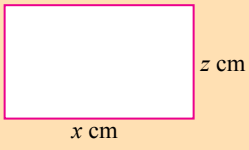
In Form 3, you have learnt the calculation for simple interest, I that involves P, r and t. The situation above shows an example of joint variation. A **joint variation** is a direct variation in which one variable varies as a product of two or more variables.

MIND MOBILISATION 4 Individual

Aim: To determine the relation between three variables for a joint variation.

Steps:

1. The diagram on the right shows a rectangle with x cm in length and z cm in width. It is given that the area of the rectangle is y cm². Complete the table below.



(A) If z is a constant				
Length, x (cm)	2	3	4	5
Width, z (cm)	6	6	6	6
Area, y (cm²)				

(B) If x is a constant				
Length, x (cm)	2	2	2	2
Width, z (cm)	6	5	4	3
Area, y (cm²)				

(C) If x, z and y are variables				
Length, x (cm)	2	3	4	5
Width, z (cm)	8	6	3	2
Area, y (cm²)				
$\frac{y}{xz}$				

Discussion:

1. What is the relation between y and x if z is a constant?

2. What is the relation between y and z if x is a constant?

3. What is the relation between y, x and z if three of them are variables?

The results of Mind Mobilisation 4 show that y varies directly as x, and y varies directly as z. Hence, y varies jointly as x and z, that is $y \propto xz$.

In general,

For a **joint variation**, y varies jointly as x^m and z^n can be written as

$y \propto x^m z^n$ (variation relation)

$y = kx^m z^n$ (equation form)

where

$m = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$,

$n = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$ and

k is a constant.

Smart TiPS

$y \propto xz$ is the combination of two relations $y \propto x$ and $y \propto z$.

Critical Mind

Discuss the relation of the equation of simple interest, $I = Prt$.

Smart TiPS

$y \propto xz$ can be read as
- y varies directly as x and z
- y varies jointly as x and z
- y is jointly proportional to x and z

Example 9

Write the relation using the symbol $\propto$ and in equation form for each of the following.

(a) p varies directly as q and $\sqrt{r}$.

(b) y varies directly as the square of w and the cube of x.

(c) The volume of a prism, V varies directly as the area of cross-section, A and its height, h.

(d) The mass, w of a cylindrical metal rod varies directly as its length, p and the square of the diameter of its base, d.

Solution:

(a) $p \propto q\sqrt{r}$ ← $p \propto q$ and $p \propto \sqrt{r}$

$p = kq\sqrt{r}$

(b) $y \propto w^2x^3$ ← $y \propto w^2$ and $y \propto x^3$

$y = kw^2x^3$

(c) $V \propto Ah$ ← $V \propto A$ and $V \propto h$

$V = kAh$

(d) $w \propto pd^2$ ← $w \propto p$ and $w \propto d^2$

$w = kpd^2$

Example 10

Given that $x \propto y^2z$, express x in terms of y and z if $x = 6$ when $y = 3$ and $z = 5$.

Solution:

$x \propto y^2z$

$x = ky^2z$ ← Write the relation in the form of equation.

$6 = k(3)^2(5)$ ← Substitute the values of x, y and z into the equation to obtain the value of k.

$k = \frac{6}{(3)^2(5)}$

$= \frac{2}{15}$

Hence, $x = \frac{2}{15}y^2z$

Example 11

The gravitational potential energy, E Joules, of an object varies directly as its mass, m kg, the gravitational acceleration, g m s^{-2} and its position at a height of h m from the ground. Given that $E = 197$ Joules when $m = 4$ kg, $g = 9.81$ m s^{-2} and $h = 5$ m, write an equation that relates E, m, g and h.

Solution:

$E \propto mgh$

$E = kmgh$ ← Write the relation in the form of equation.

$197 = k(4)(9.81)(5)$ ← Substitute the values of E, m, g and h into the equation to obtain the value of k.

$$k = \frac{197}{(4)(9.81)(5)}$$

$= 1$

Hence, $E = mgh$

Info Bulletin

(i) Gravitational potential energy is the energy stored within an object, due to the object's position.

(ii) The gravitational acceleration, g, is a constant. The energy is stored due to Earth's gravitational pull towards an object. The value of g differs between celestial bodies. For example, the value of g on Earth is 9.81 m s^{-2} and on the moon is 1.62 m s^{-2}.

Example 12

Three quantities, S, T and U vary as shown in the table on the right. It is given that S varies directly as T and the cube root of U. Calculate the values of x and y.

S	6	x	50
T	0.8	1.2	40
U	27	125	y

Solution:

$S \propto T\sqrt[3]{U}$

$S = kT\sqrt[3]{U}$

$S = 6$ when $T = 0.8$ and $U = 27$,

$6 = k(0.8)(\sqrt[3]{27})$

$$k = \frac{6}{(0.8)(\sqrt[3]{27})}$$

$= 2.5$

Hence, $S = 2.5T\sqrt[3]{U}$

When $T = 1.2$ and $U = 125$,

$x = 2.5(1.2)(\sqrt[3]{125})$

$= 15$

When $S = 50$ and $T = 40$,

$50 = 2.5(40)(\sqrt[3]{y})$

$$\sqrt[3]{y} = \frac{50}{(2.5)(40)}$$

$= 0.5$

$y = 0.5^3$

$= 0.125$

Alternative Method:

Using the concept of proportion:

Given that $S_1 = 6$, $T_1 = 0.8$, $U_1 = 27$ and $S_2 = x$, $T_2 = 1.2$, $U_2 = 125$

$$\frac{S_1}{T_1\sqrt[3]{U_1}} = \frac{S_2}{T_2\sqrt[3]{U_2}}$$

$$\frac{6}{(0.8)(\sqrt[3]{27})} = \frac{x}{(1.2)(\sqrt[3]{125})}$$

$$x = \frac{(6)(1.2)(\sqrt[3]{125})}{(0.8)(\sqrt[3]{27})}$$

$= 15$

Given that $S_1 = 6$, $T_1 = 0.8$, $U_1 = 27$ and $S_2 = 50$, $T_2 = 40$, $U_2 = y$

$$\frac{S_1}{T_1\sqrt[3]{U_1}} = \frac{S_2}{T_2\sqrt[3]{U_2}}$$

$$\frac{6}{(0.8)(\sqrt[3]{27})} = \frac{50}{(40)(\sqrt[3]{y})}$$

$$\sqrt[3]{y} = \frac{(50)(0.8)(\sqrt[3]{27})}{(40)(6)}$$

$= 0.5$

$y = 0.5^3$

$= 0.125$

Self Practice 1.1c

1. Write the relation by using the symbol $\propto$ for each of the following.
(a) s varies directly as t and u.
(b) v varies directly as w^2 and x.
(c) a varies directly as the cube of b and the square root of c.
(d) The area of the curved surface, A cm^2, of a cylinder varies directly as the radius of its base, r cm and its height, h cm.

2. Calculate the constant, k for each of the following.

(a) p varies directly as q^3 and $r^{\frac{1}{3}}$. Given that $p = 5.184$ when $q = 1.2$ and $r = 216$.

(b) p varies directly as q, r and the square of s. Given that $p = \frac{1}{3}$ when $q = \frac{1}{5}$, $r = \frac{3}{2}$ and $s = \frac{1}{3}$.

3. Given that $y = 6$ when $x = 0.64$ and $z = 5$, express y in terms of x and z if
(a) y varies directly as $\sqrt{x}$ and z,
(b) y varies directly as x and the square of z.

4. The price of a metal rod, RMx, varies directly as the length, p cm and the square of the radius, r cm. If a metal rod with a length of 150 cm and a radius of 3 cm is sold at the price of RM27, write an equation that relates x with p and r.

5. It is given that G varies directly as H and the square root of M. If $G = 42$ when $H = 7$ and $M = 16$, calculate
(a) the value of G when $H = 4$ and $M = 81$,
(b) the value of M when $G = 18$ and $H = 20$.

6. The table below shows the changes in three quantities. It is given that P varies directly as the cube of Q, and R. Calculate the values of x and y.

P	86.4	x	1.215
Q	1.2	2	y
R	10	0.4	9

7. The kinetic energy, E joules, of an object varies directly as the mass, w kg and the square of speed, v m s^{-1}, of the object. It is given that the kinetic energy of a 3-kg mass object moves with a speed of 12 m s^{-1} is 216 Joules. Calculate the speed in m s^{-1}, of an object if the mass and the kinetic energy are 5 kg and 640 Joules respectively.

8. The volume of a cone, V cm^3, varies directly as the height, h cm and the square of its base radius, r cm. A cone with a height of 21 cm and a radius of 6 cm has a volume of 792 cm^3. Calculate the volume in cm^3, of a cone with a height of 14 cm and a radius of 15 cm.

How to solve problems involving direct variation?

Learning Standard

Solve problems involving direct variation.

Historical Treasure

The relation between the volume and the temperature of a fixed mass of gas at a fixed pressure is first researched by Jacques Charles.

Example 13

Charles' Law states that for a fixed mass of gas, the volume, V cm^3, of the gas is directly proportional to its absolute temperature, T Kelvin, if the pressure of the gas is fixed. It is given that a container contains 30 cm^3 of gas at a temperature of 30°C.

(a) Express V in terms of T.

(b) Calculate the volume in cm^3, of the gas if the temperature changes to –11°C.
[Formula for the conversion of temperature in degree Celsius to Kelvin: x°C = $(273 + x)$ K]

Solution:

Understanding the problem

$V \propto T$

$V = 30$ when $T = (273 + x)$K

Devising a strategy

(a) Write the direct variation in the form of equation.

(b) Substitute the value of T into the equation and hence, calculate the volume of the gas.

Making a conclusion

(a) $V = \frac{10}{101}T$

(b) $V = 25.94$ cm^3

Implementing the strategy

(a) $V \propto T$

$V = kT$

$V = k(273 + x)$

$30 = k(273 + 30)$

$k = \frac{30}{303}$

$= \frac{10}{101}$

$V = \frac{10}{101}T$

(b) $V = \frac{10}{101}T$

$V = \frac{10}{101}(273 - 11)$

$= 25.94$ cm^3

Example 14

Madam Soon saves her money in a savings account. It is given that the interest, I, received varies directly as the principal, p, and the period in year, t of the savings. Madam Soon receives an interest of RM200 when she saves RM4 000 for two years.

(a) Calculate the period of the savings for Madam Soon to receive an interest of RM650 with a principal of RM5 200.

(b) Madam Soon wants to get the same amount of interest but reduce the period of her savings in (a). Does she need to increase or decrease her principal? Explain your answer.

Solution:

Understanding the problem

$I \propto pt$

$I = 200$ when $p = 4\ 000$ and $t = 2$

Devising a strategy

(a) Write the direct variation in the form of equation and calculate the value of t when $I = 650$ and $p = 5\ 200$.
(b) Understand the changes for every variable in the equation.

Making a conclusion

(a) 5 years.
(b) The principal has to be increased to get the same amount of interest if the period of the savings is reduced. This is because the interest received is directly proportional to the product of the principal and the period.

Implementing the strategy

(a) $I = kpt$

$200 = k(4\ 000)(2)$

$k = \dfrac{200}{(4\ 000)(2)}$

$= 0.025$

$\therefore I = 0.025pt$

When $I = 650$ and $p = 5\ 200$,

$650 = (0.025)(5\ 200)t$

$t = \dfrac{650}{(0.025)(5\ 200)}$

$= 5$ years

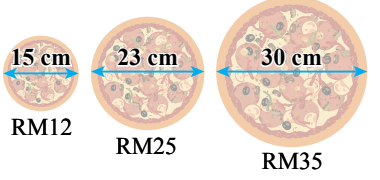
(b) To maintain the value of I in the equation $I = 0.025pt$,
- when p decreases, t increases
- when p increases, t decreases

Self Practice 1.1d

1. Lee fills a tank with water using a rubber hose at 8:00 a.m. At 11:00 a.m., Lee discovers that the tank is filled with 48% of water.
(a) Write an equation that relates the volume of water, V, that is filled into the tank with the time taken, t.
(b) At what time will the tank be fully filled with water?

2. Aminah wants to cut a few triangles from a piece of card. It is given that the area of the triangles cut, A cm^2, varies directly as its base, x cm, and its height, y cm. At the beginning, she cuts a triangle with $A = 14$, $x = 7$ and $y = 4$.
(a) Write the relation between A with x and y.
(b) Aminah plans to cut the second triangle with the value of base increased by 20% and the value of height decreased by 10%. What is the percentage of changes in the area for the second triangle?

3. A pizza shop sells three sizes of pizza at different prices as shown in the diagram on the right. Does the price of pizza, RMp, vary directly as the surface area, A cm^2, of the pizza? If it does not, which size of pizza is worth buying?



1.2 Inverse Variation

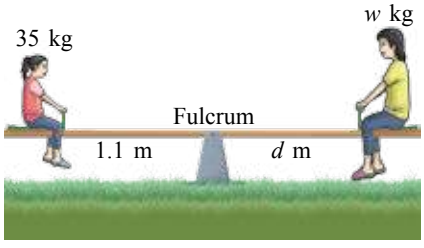
What is the meaning of inverse variation?

Learning Standard

Explain the meaning of inverse variation.

While on a seesaw, the heavier person will be at a lower position whereas the lighter person will be at a higher position. Do you know how to balance the seesaw?

To balance a seesaw, the heavier one has to sit closer to the fulcrum of the seesaw, or the lighter one has to sit further away from the fulcrum. Observe the diagram on the right. The relation between the person's mass, w, and the distance between the person and the fulcrum, d, is said to be varied inversely. When w increases, d decreases, and vice versa.



MIND MOBILISATION 5 Individual

Aim: To explain the meaning of inverse variation.

Steps:

1. Understand the following situation and then answer questions in the Discussion.
   A school hall has a few doors. The table below shows the relation between the number of doors opened, x, and the time taken, y, for a group of pupils to leave the hall.

Number of doors opened, x	2	3	4	5	6
Time taken, y (minutes)	24	16	12	9.6	8

Discussion:

1. If the number of doors opened increases, does the time taken for the pupils to leave the hall increase or decrease?
2. If the number of doors opened decreases, does the time taken for the pupils to leave the hall increase or decrease?
3. What is the relation between the number of doors opened and the time taken for the pupils to leave the hall?

The results of Mind Mobilisation 5 show that when the number of doors opened increases, the time taken for the pupils to leave the hall decreases. Similarly, when the number of doors opened decreases, the time taken for the pupils to leave the hall increases. The change in the number of doors opened leads to an opposite implication to the time taken for the pupils to leave the hall.

This relation is known as inverse variation. In general,

In **inverse variation**, variable y increases when the variable x decreases at the same rate, and vice versa. This relation can be written as y varies inversely as x.

Example 15

Chia Ming takes part in a 42-km marathon. The table below shows the relation between the time taken by Chia Ming and his average speed.

Time, t (hours)	4	5	6	7	8
Average speed, v (km/h)	10.50	8.40	7.00	6.00	5.25

State the change in average speed if the time taken
(a) increases by two times,
(b) decreases by 1.5 times.

Solution:

(a)

t	v
4	10.50
8	5.25

$\times 2$ (4 → 8), $\div 2$ (10.50 → 5.25)

When the time increases two times, then the average speed decreases two times.

(b)

t	v
6	7.00
4	10.50

$\div 1.5$ (6 → 4), $\times 1.5$ (7.00 → 10.50)

When the time decreases 1.5 times, then the average speed increases 1.5 times.

Self Practice 1.2a

1. The table on the right shows the relation between the number of workers and the number of days needed to install tiles in a house. State the change in the number of days if the number of workers
(a) is multiplied by two,
(b) decreases by half.

Number of workers	Number of days
2	12
4	6
6	4
8	3

2. Cikgu Farid has a sum of money to buy gifts for the winners of a Mathematics quiz. If the price of a gift is RM10, then Cikgu Farid can buy 10 gifts. State the number of gifts that can be bought if the price of a gift
(a) increases by two times,
(b) decreases by 50%.

What is the relation between two variables for an inverse variation?

Learning Standard

Determine the relation between two variables for an inverse variation.

MIND MOBILISATION 6 Group

Aim: To determine the relation between two variables for an inverse variation.

Steps:

A piece of cardboard has an area of 1.44 m². The cardboard will be cut into y small square cards with an area of x m² each.

1. Open the worksheet by scanning the QR code to do this activity.

	A	B	C	D	E	F
1	Area of cardboard =	1.44 m²				
2						
3	Area of small square card, x (m²)	0.02				
4	Number of small square cards, y	72				
5	xy	1.44				
6	$\frac{1}{x}$	50				
7						

Scan the QR code or visit bit.do/WSChap1 to obtain the worksheet.

2. Insert the area of the small card, $x = 0.04$ in coloured cell and click Enter. Repeat the same step for the value of x equals to 0.09, 0.16 and 0.36. Observe the values of y, xy and $\frac{1}{x}$ for each value of x.
3. Draw the graph of y against x and the graph of y against $\frac{1}{x}$ using a suitable scale based on the obtained values.

Discussion:

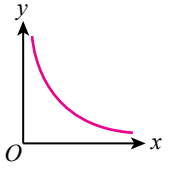
1. Compare the values of xy. What conclusion can be made on the values of xy?
2. What are the shapes of the graphs obtained?
3. What is the relation between y and x?

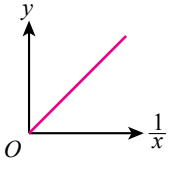
The results of Mind Mobilisation 6 show that for an inverse variation,

(a) the value of xy is a constant. Hence, $k = xy$,

(b) the graph of y against x is a hyperbola and the graph of y against $\frac{1}{x}$ is a straight line that starts from the origin,

(c) y varies inversely as x.





When variable y varies inversely as variable x, the value of xy is a constant that is represented with k.

Relation between the value of k and the concept of proportion:

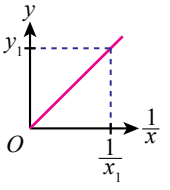
y	y_1	y_2	y_3	y_4
x	x_1	x_2	x_3	x_4

$k = x_1y_1 = x_2y_2 = x_3y_3 = x_4y_4$

Relation between the value of k and the gradient of the straight line that starts from the origin for the graph of y against $\frac{1}{x}$:

Gradient,

$$m = \frac{y_1}{\frac{1}{x_1}}$$
$$= x_1y_1$$
$$= k$$



This relation is true only if the straight line starts from the origin.

Smart TiPS

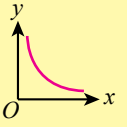
The relation of inverse variation written as 'y varies inversely as x' can also be written as 'x and y are inversely proportional'.

Hence, we can relate the two variables x and y in the form of equation, that is

$$xy = k$$
$$y = \frac{k}{x}.$$

For an **inverse variation**, y varies inversely as x can be written as

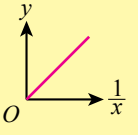
$$y \propto \frac{1}{x} \text{ (variation relation)}$$
$$y = \frac{k}{x} \text{ (equation form)}$$



where k is a constant.

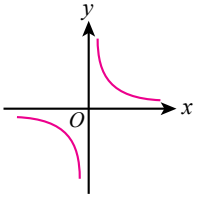
(a) The graph of y against x is a hyperbola.

(b) The graph of y against $\frac{1}{x}$ is a straight line that starts from the origin ($x \neq 0$).



MEMORY BOX

$y = \frac{a}{x}$, $x \neq 0$ is a reciprocal function with a hyperbola graph.



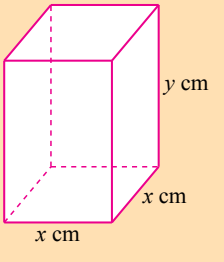
MIND MOBILISATION 7 Group

Aim: To determine the relation between the variables y and x^2 for an inverse variation.

Steps:

1. It is given that the volume of a cuboid with a square base is 180 cm^3. The table below shows the relation between the length of sides of the base and the height of the cuboid. Complete the table below.

Length of sides of the base, x (cm)	2	3	4	5	6
Height, y (cm)					
xy					
x^2y					
$\frac{1}{x^2}$					



2. Draw the graph of y against x and the graph of y against $\frac{1}{x^2}$ using a suitable scale based on the table.

Discussion:

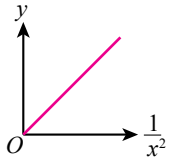
1. Compare the values of xy and x^2y. What conclusion can be made?
2. Which graph is a straight line graph that starts from the origin?
3. What is the relation between y and x^2?

The results of Mind Mobilisation 7 show that for an inverse variation,

(a) the value of x^2y is a constant. Hence, $k = x^2y$,

(b) the graph of y against $\frac{1}{x^2}$ is a straight line that starts from the origin,

(c) y varies inversely as x^2.



In general,

For an **inverse variation**, y varies inversely as x^n can be written as

$y \propto \frac{1}{x^n}$ (variation relation)

$y = \frac{k}{x^n}$ (equation form)

where $n = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$ and k is a constant.

The graph of y against $\frac{1}{x^n}$ is a straight line that starts from the origin ($x \neq 0$) where k is the gradient of the straight line.

Interactive Platform

When $y = \frac{1}{x^2}$, y does not vary directly as x. Discuss this statement.

Smart TiPS

$y \propto \frac{1}{x}$ is true if and only if $n = 1$.

Example 16

It is given that the time needed, t, to assemble a piece of furniture varies inversely as the number of workers, x. The table below shows the relation between x and t.

Number of workers, x	2	3	4	5	6
Time needed, t (minutes)	180	120	90	72	60

By calculating the values of xt and x^2t, determine whether t varies inversely as x or x^2. Hence, write the relation by using the symbol $\propto$.

Solution:

x	2	3	4	5	6
t	180	120	90	72	60
xt	$2(180) = 360$	$3(120) = 360$	$4(90) = 360$	$5(72) = 360$	$6(60) = 360$
x^2t	$2^2(180) = 720$	$3^2(120) = 1080$	$4^2(90) = 1440$	$5^2(72) = 1800$	$6^2(60) = 2160$

The value of xt is a constant, whereas the value of x^2t is not a constant. Hence, t varies inversely as x, that is $t \propto \frac{1}{x}$.

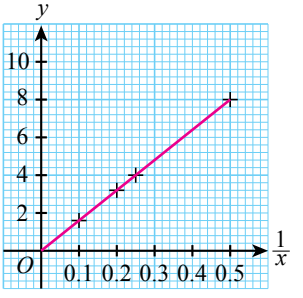
Example 17

Two quantities, x and y, vary based on the table on the right. By using a suitable scale, draw the graph of y against $\frac{1}{x}$ and show that y varies inversely as x.

x	2	4	5	10
y	8	4	3.2	1.6

Solution:

$\frac{1}{x}$	0.5	0.25	0.2	0.1
y	8	4	3.2	1.6



The graph of y against $\frac{1}{x}$ shows a straight line that starts from the origin. Hence, y varies inversely as x.

Smart TIPS

(i) y varies inversely as x,
$$y \propto \frac{1}{x}$$

(ii) y varies inversely as the square of x,
$$y \propto \frac{1}{x^2}$$

(iii) y varies inversely as the cube of x,
$$y \propto \frac{1}{x^3}$$

(iv) y varies inversely as the square root of x,
$$y \propto \frac{1}{\sqrt{x}}$$

(v) y varies inversely as the cube root of x,
$$y \propto \frac{1}{\sqrt[3]{x}}$$

Example 18

It is given that $x = 0.25$ when $y = 3$. Express y in terms of x if
(a) y varies inversely as x,
(b) y varies inversely as the square root of x.

Solution:

(a) $y \propto \frac{1}{x}$

$y = \frac{k}{x}$ ← Write the relation in the form of equation.

$3 = \frac{k}{0.25}$ ← Substitute the values of y and x into the equation to obtain the value of k.

$k = 3(0.25)$
$= 0.75$

Hence, $y = \frac{0.75}{x}$

(b) $y \propto \frac{1}{\sqrt{x}}$

$y = \frac{k}{\sqrt{x}}$

$3 = \frac{k}{\sqrt{0.25}}$

$k = 3(\sqrt{0.25})$
$= 1.5$

Hence, $y = \frac{1.5}{\sqrt{x}}$

Example 19

The force of gravity, F, varies inversely as the square of the distance between two objects, d. It is given that the force of gravity between two objects is 15 N when the distance between them is 1.2 cm. Write an expression of F in terms of d.

Solution:

$$F \propto \frac{1}{d^2}$$

$$F = \frac{k}{d^2}$$ ← Write the relation between F and d in the form of equation.

$$15 = \frac{k}{(1.2)^2}$$ ← Substitute the values of F and d into the equation to obtain the value of k.

$$k = 15(1.2)^2$$
$$= 21.6$$

Hence, $F = \frac{21.6}{d^2}$

Historical Treasure

Sir Isaac Newton (1642-1727), a famous English physicist, published Newton's Laws of Gravity in his book Philosophiae Naturalis Principia Mathematica in 1687.

Example 20

It is given that p varies inversely as q. If $p = 2$ when $q = 7$, calculate the value of p when $q = 1.6$.

Solution:

$$p \propto \frac{1}{q}$$

$$p = \frac{k}{q}$$

$$2 = \frac{k}{7}$$

$$k = 2(7)$$
$$= 14$$

Hence, $p = \frac{14}{q}$

When $q = 1.6$,

$$p = \frac{14}{1.6}$$
$$= 8.75$$

Alternative Method:

Using the concept of proportion:
Given that $p_1 = 2$, $q_1 = 7$, $q_2 = 1.6$

$$p_1 q_1 = p_2 q_2$$
$$2 \times 7 = p_2 \times 1.6$$
$$p_2 = \frac{2 \times 7}{1.6}$$
$$= 8.75$$

Self Practice 1.2b

1. The table below shows the values of two variables, x and y.

(a) Determine whether y varies inversely as x. If yes, write the relation in the form of variation.

x	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
y	6	4	3	2	1

(b) Determine whether y varies inversely as x^2. If yes, write the relation in the form of variation.

x	1	2	3	4	5
y	3.6	0.9	0.4	0.225	0.144

2. The table below shows the current, I (Ampere, A) flows in a circuit with its resistance, R (Ohm, Ω).

Resistance, R (Ω)	42	35	30	21	15
Current, I (A)	5	6	7	10	14

By plotting the graph of I against $\frac{1}{R}$ using a suitable scale, determine whether I varies inversely as R.

3. It is given that $g = 0.15$ when $h = 8$. Express g in terms of h if
(a) g varies inversely as h,
(b) g varies inversely as h^2,
(c) g varies inversely as the cube root of h.

4. Given that $y = 0.5$ when $x = 16$, calculate the value of y when $x = 0.04$ if

(a) $y \propto \frac{1}{x}$

(b) $y \propto \frac{1}{x^3}$

(c) $y \propto \frac{1}{x^{\frac{1}{2}}}$

(d) $y \propto \frac{1}{\sqrt[3]{x}}$

5. Calculate the values of s and t for each of the following relations.

(a) y varies inversely as x.

x	3	s	5
y	2	1.5	t

(b) y varies inversely as the square root of x.

x	$\frac{9}{4}$	s	0.04
y	8	2.4	t

6. A company prints a certain number of books every day. The table below shows the number of printing machines in operation, M and the time needed, T to print the books. It is given that T varies inversely as M.

M	6	8	q
T	10	p	4

(a) Express T in terms of M.
(b) Determine the values of p and q.

7. The period of oscillation, T, of a simple pendulum varies inversely as the square root of gravitational acceleration, g. In an experiment, the period of oscillation is 1.01 seconds when the gravitational acceleration is 9.85 m s^{-2}. Express T in terms of g.

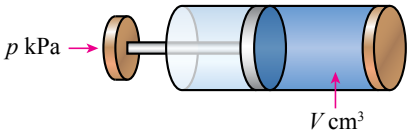
How to solve problems involving inverse variation?

Learning Standard

Solve problems involving inverse variation.

Example 21

According to Boyle's Law, the gas pressure, p, of a mass of gas is inversely proportional to the volume of the gas, V, if the temperature of the gas is fixed. The diagram on the right shows the gas trapped in a cylinder. When the volume inside the cylinder is 80 cm³, the gas pressure is 190.25 kPa. Calculate the volume of the gas in cm³, when the gas pressure inside the cylinder is 121.76 kPa.



Solution:

Understanding the problem

- p is inversely proportional to V
- When $V = 80$ cm³, $p = 190.25$ kPa

Devising a strategy

Determine the relation between p and V in the form of equation. Hence, calculate the value of V when $p = 121.76$ kPa.

Implementing the strategy

$$p \propto \frac{1}{V}$$

$$p = \frac{k}{V}$$

When $V = 80$, $p = 190.25$,

$$190.25 = \frac{k}{80}$$

$$k = 190.25 \times 80$$

$$= 15\ 220$$

Hence, $p = \dfrac{15\ 220}{V}$

When $p = 121.76$,

$$121.76 = \frac{15\ 220}{V}$$

$$V = \frac{15\ 220}{121.76}$$

$$= 125 \text{ cm}^3$$

Making a conclusion

When $p = 121.76$ kPa, $V = 125$ cm³

Self Practice 1.2c

1. The table on the right shows the resistance of a length of wire varies as the radius of its cross-section.

Resistance, R (Ω)	1	4	9	16
Radius of cross-section, r (mm)	1.2	0.6	0.4	0.3

(a) Determine whether the resistance, R, of the wire varies inversely as r^2.

(b) Calculate the radius in mm, of the wire if the resistance is 25 Ω.

2. The number of cubes, b, produced from a fixed quantity of metal varies inversely as the cube of its sides, p cm. If $b = 16$ when $p = 1.5$, calculate the value of p when $b = 250$.

3. The number of oscillations, A of a simple pendulum varies inversely as the square root of the length of the pendulum, p cm, in a fixed period. Given that the number of oscillations is 9 when the length of the pendulum is 36 cm, calculate the length of the simple pendulum if the number of oscillations is 15.

1.3 Combined Variation

What is the relation between three or more variables for a combined variation?

Learning Standard

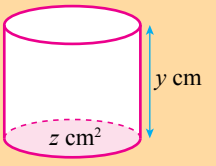
Determine the relation between three or more variables for a combined variation.

MIND MOBILISATION 8 Group

Aim: To determine the relation between three variables for a combined variation which involves direct variation and inverse variation.

Steps:

1. The formula for the volume of a cylinder is the product of its base area and its height. If x, y and z represent the volume, height and base area of the cylinder respectively, complete the table below.



(A) If z is fixed				
Volume, x (cm^3)	100	200	300	400
Height, y (cm)				
Base area, z (cm^2)	50	50	50	50
$\frac{x}{y}$				

(B) If x is fixed				
Volume, x (cm^3)	120	120	120	120
Height, y (cm)				
Base area, z (cm^2)	40	60	80	100
yz				

(C) If x, y and z are variables				
Volume, x (cm^3)	80	180	320	500
Height, y (cm)				
Base area, z (cm^2)	20	30	40	50
$\frac{yz}{x}$				

Discussion:

1. What is the value of $\frac{x}{y}$? Hence, state the relation between y and x if z is a constant.

2. What is the value of yz? Hence, state the relation between y and z if x is a constant.

3. What is the value of $\frac{yz}{x}$? Hence, state the relation between y, x and z if three of them are variables.

The results of Mind Mobilisation 8 show that y varies directly as x, and y varies inversely as z. Therefore, y varies directly as x and inversely as z, that is $y \propto \frac{x}{z}$.

In general,

For a **combined variation**, y varies directly as x^m and inversely as z^n can be written as

$y \propto \frac{x^m}{z^n}$ (variation relation)

$y = \frac{kx^m}{z^n}$ (equation form)

where

$m = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$, $n = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$ and

k is a constant.

Example 22

It is given that y varies directly as the square of x and inversely as the square root of z. If $y = 8$ when $x = 4$ and $z = 36$, express y in terms of x and z.

Solution:

$$y \propto \frac{x^2}{\sqrt{z}}$$

$$y = \frac{kx^2}{\sqrt{z}}$$ ← Write the relation between y, x and z in the form of equation.

$$8 = \frac{k(4)^2}{\sqrt{36}}$$ ← Substitute the values of y, x and z into the equation to obtain the value of k.

$$k = \frac{8\sqrt{36}}{4^2}$$

$$= 3$$

Hence, $y = \frac{3x^2}{\sqrt{z}}$

Smart TiPS

(i) y varies directly as w and the cube of x, and inversely as the cube root of z,

$$y \propto \frac{wx^3}{\sqrt[3]{z}}$$

(ii) y varies directly as the product of w^2 and $\sqrt{x}$, and inversely as z^3,

$$y \propto \frac{w^2\sqrt{x}}{z^3}$$

Example 23

The table below shows the values of P, Q and R. It is given that P varies directly as the cube of Q and inversely as R. Calculate the values of x and y.

P	4	3.6	0.081
Q	2	6	y
R	0.6	x	2.7

Solution:

$P \propto \frac{Q^3}{R}$

$P = \frac{kQ^3}{R}$

$4 = \frac{k(2)^3}{0.6}$

$k = \frac{4(0.6)}{2^3} = \frac{3}{10}$

Hence, $P = \frac{3Q^3}{10R}$

When $P = 3.6$, $Q = 6$ and $R = x$,

$P = \frac{3Q^3}{10R}$

$3.6 = \frac{3(6)^3}{10x}$

$x = \frac{3(6)^3}{10(3.6)}$

$= 18$

When $P = 0.081$, $Q = y$ and $R = 2.7$,

$P = \frac{3Q^3}{10R}$

$0.081 = \frac{3(y)^3}{10(2.7)}$

$y = \sqrt[3]{\frac{0.081(10)(2.7)}{3}}$

$= 0.9$

Self Practice 1.3a

1. Write in the form of variation and in the form of equation for each of the following combined variations.
(a) w varies directly as the cube root of v and inversely as the square of x.
(b) F varies directly as G and H^3, and inversely as $\sqrt{t}$.
(c) The acceleration of an object, A varies directly as the distance travelled, s, and inversely as the square of the time taken, t, by the object.

2. The time taken, t hours, to arrange chairs in a hall varies directly as the number of chairs, c, and inversely as the number of workers involved, p. It is given that 5 workers used 2 hours to arrange 1 000 chairs. Express t in terms of c and p.

3. It is given that M varies directly as the square of N and inversely as P. If $M = 4.8$ when $N = 6$ and $P = 1.5$, calculate
(a) the value of P when $M = 0.8$ and $N = 2.4$,
(b) the value of N when $M = 19$ and $P = 3.8$.

4. The table on the right shows the changes in three quantities T, e and f. It is given that T varies inversely as the square root of e and the cube of f. Calculate the values of a and b.

T	5	a	0.256
e	1.44	36	b
f	2	0.4	5

How to solve problems involving combined variation?

Learning Standard

Solve problems involving combined variation.

Example 24

The pressure, p N m^{-2}, on the tyre of a wheelbarrow varies directly as the mass of the wheelbarrow, m kg, and inversely as the surface area of the tyre touching the ground, l m^2. It is given that the pressure on the tyre is 45 000 N m^{-2} when the mass of the wheelbarrow is 90 kg and the surface area of the tyre touching the ground is 0.02 m^2.
(a) Calculate the value of p when $m = 120$ and $l = 0.5$.
(b) What can be done to reduce the pressure on the tyre if the mass of the wheelbarrow is fixed?

Solution:

Understanding the problem

- p varies directly as m and inversely as l
- When $m = 90$ and $l = 0.02$, $p = 45\ 000$

Devising a strategy

(a) Determine the relation between p, m and l in the form of equation. Hence, substitute the values of $m = 120$ and $l = 0.5$ into the equation to calculate the value of p.

(b) Understand the changes of every variable in the equation.

Implementing the strategy

(a) $p \propto \frac{m}{l}$

$p = \frac{km}{l}$

$45\ 000 = \frac{k(90)}{0.02}$

$k = \frac{(0.02)(45\ 000)}{90}$

$= 10$

Hence, $p = \frac{10m}{l}$

When $m = 120$, $l = 0.5$,

$p = \frac{(10)(120)}{0.5}$

$= 2\ 400\ \text{N m}^{-2}$

(b) If the value of m is fixed,

- p decreases, when l increases
- p increases, when l decreases

Making a conclusion

(a) $p = 2\ 400\ \text{N m}^{-2}$

(b) Use a broader tyre because the pressure on the tyre reduces when the surface area of the tyre touching the ground increases, and the mass is fixed.

Self Practice 1.3b

1. Mr Kamal wants to install rectangular tiles in his bedroom. The number of tiles needed, J, varies inversely as the length, p m, and width, l m, of the tiles used. Mr Kamal needs 120 pieces of tiles if the tile is 0.4 m in length and 0.5 m in width.

(a) Calculate the number of tiles needed if the length is 0.2 m and the width is 0.3 m.

(b) If the area of the tile increases, what is the change in the number of tiles needed?

2. The average number of daily phone calls, C, between two cities varies directly as the populations of the two cities, P_1 and P_2, and inversely as the square of the distance, j, between the two cities. The distance between city A and city B is 210 km. The average number of daily phone calls between the two cities is 15 750 and the populations of city A and city B are 105 000 and 220 500 respectively. Give your answer to the nearest whole number, calculate

(a) the distance between city P and city Q if the populations are 83 400 and 62 000 respectively and the average number of daily phone calls is 19 151,

(b) the population of city J if the population of city K is 1 100 000 with the distance between the two cities is 351 km. The average number of daily phone calls between city J and city K is 18 857.

Summary Arena

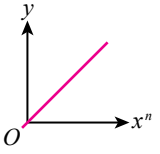
VARIATION

Direct Variation

y varies directly as x^n, for

$n = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$

- $y \propto x^n$
- $y = kx^n$ where k is a constant and $k = \frac{y}{x^n}$
- The graph of y against x^n is a straight line that passes through the origin

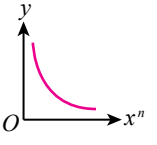


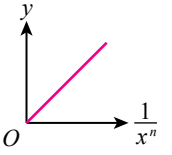
Inverse Variation

y varies inversely as x^n, for

$n = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$

- $y \propto \frac{1}{x^n}$
- $y = \frac{k}{x^n}$ where k is a constant and $k = yx^n$
- The graph of y against $\frac{1}{x^n}$ is a straight line that starts from the origin ($x \neq 0$).





y varies inversely as x^n

Joint Variation

A direct variation where a variable varies as the product of two or more variables.

- y varies jointly as x^m and z^n, for

  $m = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$ and

  $n = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$
- $y \propto x^m z^n$
- $y = kx^m z^n$ where k is a constant and $k = \frac{y}{x^m z^n}$

Combined Variation

Involves a combination of direct variation or joint variation, and inverse variation.

- For $p = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$, $q = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$ and $r = 1, 2, 3, \frac{1}{2}, \frac{1}{3}$

  (i) $y \propto \frac{x^p}{z^q}$

  $y = \frac{kx^p}{z^q}$

  (ii) $y \propto \frac{w^p x^q}{z^r}$

  $y = \frac{kw^p x^q}{z^r}$

  where k is a constant.

Reflection

At the end of this chapter, I can

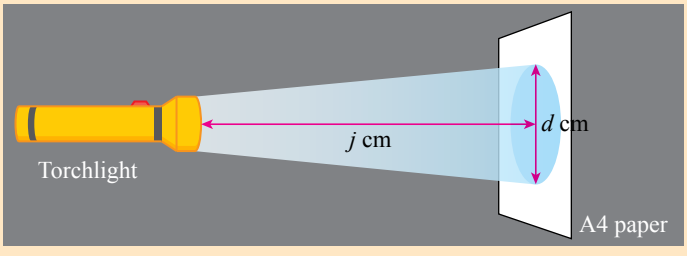
explain the meaning of direct variation.		
determine the relation between two variables for a direct variation.		
determine the relation between three or more variables for a given joint variation.		
solve problems involving direct variation.		
explain the meaning of inverse variation.		
determine the relation between two variables for an inverse variation.		
solve problems involving inverse variation.		
determine the relation between three or more variables for a combined variation.		
solve problems involving combined variation.		

MINI PROJECT

Materials: Torchlight, A4 paper and measurement tape

Steps:

1. Carry out this project in groups. Paste the A4 paper and place the torchlight facing the paper in a dark classroom as shown in the diagram below.



2. With a different distance of the torchlight from the A4 paper, j, for example 10 cm, 20 cm, 30 cm, …, record the diameter of the light image, d, formed on the paper. Construct a table to record your data.
3. Represent the data obtained using a suitable line graph.
4. Write a formula to show the relation between d and j. Then, make a conclusion on the diameter of the light image formed and the distance of the torchlight from the paper.
5. How can these findings be applied in your daily life?

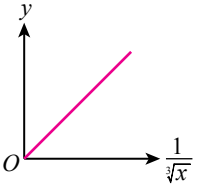
Extensive Practice

Scan the QR code or visit bit.do/QuizE01 for interactive quiz

UNDERSTAND

1. Write the variation relation for each of the following.
(a) w is directly proportional to the cube of x.
(b) a varies directly as b and inversely as the cube of c.
(c) p varies directly as q and the square root of r.
(d) The distance travelled, s m, by a bicycle varies directly as its acceleration, a m s^{-2}, and the square of the time taken, t s.

2. The diagram on the right shows the graph of y against $\frac{1}{\sqrt[3]{x}}$. Write the relation between y and $\sqrt[3]{x}$ by using the symbol $\propto$.



3. Write the following relations in sentences.
(a) $y \propto xz$ (b) $e \propto \frac{1}{f}$ (c) $p \propto \frac{\sqrt[3]{q}}{r}$ (d) $n \propto \frac{pq^2}{\sqrt{r}}$

4. State whether each of the following has a relation of y varies directly as x.
(a) $x - y = 0$ (b) $y + 3 = x$ (c) $xy = 10$ (d) $\frac{x}{y} = 0.5$

MASTERY

5. Calculate the constant, k for each of the following.
(a) L varies directly as the cube of m. $L = 16.384$ when $m = 3.2$.
(b) h varies directly as a and the square of b. $h = 96$ when $a = 18$ and $b = 4$.
(c) P varies directly as q^2 and r, and inversely as $\sqrt[3]{s}$. $P = 17.01$ when $q = 4.5$, $r = 9$ and $s = 3\ 375$.

6. It is given that m varies inversely as n and p. If $m = 6$ when $n = 0.4$ and $p = 5$, write an equation that relates m, n and p.

7. It is given that $f \propto g^2h$ and $f = 24$ when $g = 4$ and $h = 5$. Calculate the value of g when $f = 5.88$ and $h = 10$.

8. The table on the right shows the changes in three quantities. It is given that y varies directly as x and inversely as the square root of z. Calculate the values of m and n.

y	4	m	51
x	0.3	6	1.7
z	3.24	225	n

CHALLENGE

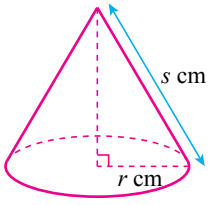
9. P varies inversely as Q and $Q = 3R - 2$. It is given that $P = 0.02$ when $R = 4$.
(a) Express P in terms of Q.
(b) Calculate the value of R when $P = 5$.

10. The electric current, I (Ampere) varies directly as the power, P (Watt) and inversely as the voltage, V (Volt) for an electrical appliance. It is given that a hairdryer with 550 W of power and 240 V of voltage uses an electric current of 2.2 A. Calculate the electric current used by a fan with 75 W of power and 240 V of voltage.

11. The area of the curved surface, A cm^2, of a cone varies directly as the radius of its base, r cm, and the slant height, s cm. It is given that $A = 88$ cm^2 when $r = 3.5$ cm and $s = 8$ cm.

(a) Calculate the value of A when $r = 5$ cm and $s = 9.8$ cm.

(b) What is the change in the area of the curved surface if the slant height decreases and the base radius is fixed?

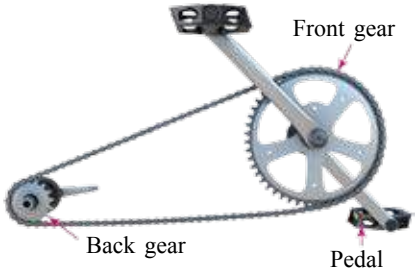


12. It is given that Y varies directly as X and inversely as W. If $Y = 0.9$ when $X = 18$ and $W = 5$, calculate

(a) the value of W when $Y = 20$ and $X = 6$,

(b) the percentage of the change of Y when X increases by 10% and W decreases by 20%.

13. The speed of a bicycle, S, varies directly as the number of revolutions per minute of the bicycle pedal, P, and the number of the front gear teeth, d, and inversely as the number of back gear teeth, b. Santhami rides a bicycle at a speed of 26.4 km per hour. The bicycle pedal moves at 75 revolutions per minute with 40 front gear teeth and 20 back gear teeth. Describe the change in speed of Santhami's bicycle if he pedals at 90 revolutions per minute.



EXPLORING MATHEMATICS

The variation concept is widely used in science. Some examples are:

Ohm's Law	Law of Refraction	Boyle's Law	Newton's Law of Universal Gravitation
$V = IR$	$\frac{\sin i}{\sin r} = \text{constant}$	$PV = \text{constant}$	$F = G\frac{m_1 m_2}{d^2}$

In groups, find information from the internet on the quantities represented by the variables for the above formulae and how the concept of variation is applied in these laws. Briefly describe the usage of the formulae. Then, conduct further investigations on the application of the concept of variation in the field of finance, environment and social science. Present your findings to share information with other groups.

CHAPTER 2 Matrices

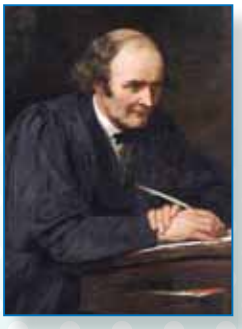
What will you learn?

- Matrices
- Basic Operation on Matrices

Why study this chapter?

Matrices are used in the field of mathematics to represent and solve problems in algebra, statistic and geometry. In daily life, it can be used by a purchasing manager to record the amount of inventory in his business.

Do you know?



Arthur Cayley (1821-1895) was an English mathematician who developed the theory of matrices from the aspect of algebra in his work A Memoir on the Theory of Matrices. Cayley created matrices when he was researching the theory of transformation. Two American mathematicians, Benjamin Peirce (1809-1880) and Charles S. Peirce (1839-1914) worked together with Cayley in the development of algebraic matrices.

For more information:

bit.do/DoYouKnowChap2

WORD BANK

matrix	*matriks*
row matrix	*matriks baris*
identity matrix	*matriks identiti*
column matrix	*matriks lajur*
square matrix	*matriks segi empat sama*
rectangular matrix	*matriks segi empat tepat*
zero matrix	*matriks sifar*
inverse matrix	*matriks songsang*
scalar multiplication	*pendaraban skalar*
determinant	*penentu*
order	*peringkat*
element	*unsur*

According to Digital Report 2018, the total number of internet users has reached 25.08 million people, that is 79% of the residents in Malaysia. Nowadays, the society prefers to carry out daily activities such as shopping and making payment for services through the internet. Data is created in this process and supports the growth of big data.

Big data refers to data that is very large, complex and difficult to process with regular database management. This set of data can help the management teams in various fields to make better decisions. In the business field, big data is analysed to reduce cost and time, develop new products and strategise business plans. The process of data analysis largely involves matrices.

2.1 Matrices

How to represent the information from real situations in the form of matrices?

Learning Standard

Represent information from real situations in the form of matrices.

Kedai Elektrik Sinar Jaya records the sales of three types of fans in a spreadsheet. The diagram below shows the in-store and online sales of the fans for the months of March to May. How can these data be arranged?

	A	B	C	D	E	F	G	H	I	J	K	L
1		The sales of the fans in March				The sales of the fans in April				The sales of the fans in May		
2		Fan				Fan				Fan		
3		Stand	Ceiling	Wall		Stand	Ceiling	Wall		Stand	Ceiling	Wall
4	In-store	16	18	11		20	15	9		15	21	10
5	Online	5	10	4		7	12	5		10	24	10
6												
7		The total sales in March and April				The difference in sales between April and May						
8		Fan				Fan						
9		Stand	Ceiling	Wall		Stand	Ceiling	Wall				
10	In-store	36	33	20		-5	6	1				
11	Online	12	22	9		3	12	5				
12												

The data for the month of March is represented in the form of table. It can also be represented in the form of matrix as shown below.

	Fan		
	Stand	Ceiling	Wall
In-store	16	18	11
Online	5	10	4

Table form

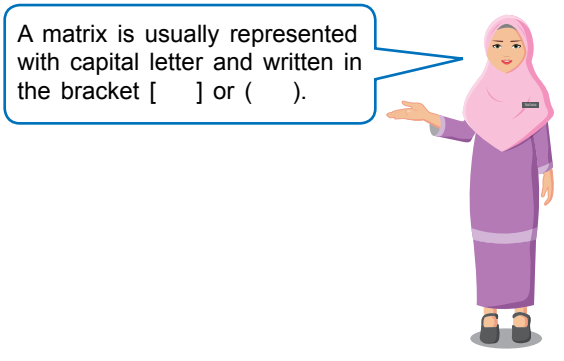
rows

$$\begin{pmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{pmatrix}$$

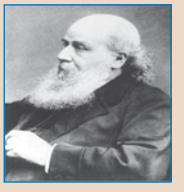
columns

Matrix form

A **matrix** is a set of numbers arranged in rows and columns to form a rectangular or a square array.



Historical Treasure



James Joseph Sylvester (1814-1897) was the first mathematician that used the term matrix in 1850.

Example 1

Represent the following information in the form of matrices.

(a) The table below shows the recommended daily calorie intake for females by category.

	Child	Teenager	Adult	Senior Citizen
Recommended daily calorie intake (kcal)	1 700	2 100	2 000	1 800

(b) The table below shows the medal collection of Malaysian contingent for three events in the 29th SEA Games.

	Gold	Silver	Bronze
Diving	13	5	1
Pencak Silat	10	2	4
Athletics	8	8	9

(c) In Test 1, Samad scores 76 marks for Bahasa Melayu, 82 marks for Mathematics and 72 marks for History, while Hamid scores 80 marks for Bahasa Melayu, 88 marks for Mathematics and 70 marks for History.

Info Bulletin

Calories are a measure of the value of energy in food. The caloric content of food depends on the content of carbohydrates, proteins and fats in it. For example, 1 g of protein equals 4 kcal.

My Malaysia

In the 29th SEA Games that was held in Kuala Lumpur in 2017, the Malaysian contingent was crowned the overall champion with 145 golds.

Critical Mind

Is [6] a square matrix? Explain your answer.

Solution:

(a) row 1 → $[1700 \quad 2100 \quad 2000 \quad 1800]$ or $\begin{bmatrix} 1700 \\ 2100 \\ 2000 \\ 1800 \end{bmatrix}$ ← column 1

This is a row matrix. A row matrix has only one row.

This is a column matrix. A column matrix has only one column.

(b) $\begin{matrix} \text{row 1} \rightarrow \\ \text{row 2} \rightarrow \\ \text{row 3} \rightarrow \end{matrix} \begin{bmatrix} 13 & 5 & 1 \\ 10 & 2 & 4 \\ 8 & 8 & 9 \end{bmatrix}$ or $\begin{bmatrix} 13 & 10 & 8 \\ 5 & 2 & 8 \\ 1 & 4 & 9 \end{bmatrix}$

column 1, column 2, column 3

These are square matrices. A square matrix has the same number of rows and columns.

(c) $\begin{matrix} \text{row 1} \rightarrow \\ \text{row 2} \rightarrow \\ \text{row 3} \rightarrow \end{matrix} \begin{bmatrix} 76 & 80 \\ 82 & 88 \\ 72 & 70 \end{bmatrix}$ or $\begin{matrix} \text{row 1} \rightarrow \\ \text{row 2} \rightarrow \end{matrix} \begin{bmatrix} 76 & 82 & 72 \\ 80 & 88 & 70 \end{bmatrix}$

column 1, column 2 / column 1, column 2, column 3

These are rectangular matrices. A rectangular matrix has different number of rows and columns.

Self Practice 2.1a

1. A fifth generation (5G) mobile technology exhibition is attended by 857 teenagers, 3 180 adults and 211 senior citizens. Represent the information in the form of matrix.

2. The table on the right shows the average Air Pollution Index (API) in Putrajaya, Jerantut and Sandakan for three days. Represent the information in the form of matrix.

	Monday	Tuesday	Wednesday
Putrajaya	53	52	50
Jerantut	20	21	20
Sandakan	47	48	46

3. The table below shows the average number of books read by the pupils in Program Nilam in SMK Setia for 2019.

	Form 3	Form 4	Form 5
Bahasa Melayu	20	18	15
English	12	10	11

Represent the information above in the form of matrix.

How to determine the order of a matrix, hence identify certain elements in a matrix?

Learning Standard

Determine the order of a matrix, hence identify certain elements in a matrix.

Order of a matrix can be determined by counting the number of rows followed by the number of columns of the matrix. For example,

row 1 → row 2 →
$$\begin{bmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix}_{2 \times 3}$$
↑ column 1 ↑ column 2 ↑ column 3

This matrix has 2 rows and 3 columns. Therefore, it is a matrix with order 2 × 3 and can be read as "matrix 2 by 3".

Matrix with m rows and n columns has the order $m \times n$ and is read as "matrix m by n".

Every number in the matrix is known as an **element** of the matrix. For example, the element at the 2nd row and 3rd column for the matrix $\begin{bmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix}$ is 4.

Capital letter is used to denote a matrix, for example $A = \begin{bmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix}$, and the element at the 2nd row and 3rd column can be represented as a_{23}, for example $a_{23} = 4$.

In general, the element at the i^{th} row and j^{th} column in matrix A can be represented as

a_{ij} — i^{th} row, j^{th} column

Hence, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

Info Bulletin

a_{21} is read as "a two one".

Info Bulletin

A row matrix has the order $1 \times n$ while a column matrix has the order $m \times 1$.

Example 2

It is given that three matrices, $P = [3 \quad -7 \quad 9]$, $Q = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 5 & -2 & 0 \\ 3 & 7 & 2 & 8 \\ -4 & 11 & 6 & 1 \\ 9 & 3 & -1 & 5 \end{bmatrix}$. Determine

(a) the order of each matrix,

(b) the element
   (i) at the first row and 3rd column of matrix P, p_{13},
   (ii) at the 2nd row and first column of matrix Q, q_{21},
   (iii) at the 3rd row and 4th column of matrix R, r_{34}.

Solution:

(a) The order of matrix P is 1×3.

The order of matrix Q is 2×1.

The order of matrix R is 4×4.

row 1 → [3 (↑ column 1) −7 (↑ column 2) 9 (↑ column 3)]

This matrix has 1 row and 3 columns.

(b) (i) p_{13} of matrix P is 9.

(ii) q_{21} of matrix Q is 5.

(iii) r_{34} of matrix R is 1.

Example 3

Given that matrix $D = \begin{bmatrix} -2 & 5 \\ 0 & 4 \\ 1 & 9 \end{bmatrix}$, determine

(a) the order of the matrix, (b) the elements d_{11}, d_{21} and d_{32}.

Solution:

(a) 3×2

(b) $d_{11} = -2$ ← d_{11} is the element at the first row and first column.

$d_{21} = 0$ ← d_{21} is the element at the 2nd row and first column.

$d_{32} = 9$ ← d_{32} is the element at the 3rd row and 2nd column.

Historical Treasure

Cuthbert Edmund Cullis (1875-1954) was an English mathematician who introduced brackets for matrices in 1913. Cullis used the notation $A = [a_{ij}]$ to represent the element at i^{th} row and j^{th} column in a matrix.

Self Practice 2.1b

1. Determine the order of the following matrices.

(a) $[15 \quad -8]$ (b) $\begin{bmatrix} 6 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & -1 & 7 \\ 8 & 0 & 2 \\ 5 & 11 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 12 & 9 & 1 \\ 5 & 10 & 7 \end{bmatrix}$

2. For each of the following matrices, determine
(i) the order of the matrix,
(ii) the element at the 2nd row and 2nd column,
(iii) the element at the 3rd row and first column.

(a) $\begin{bmatrix} 1 & 5 \\ -6 & 0 \\ 9 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 & 5 \\ -1 & 16 & 0 \\ 9 & 1 & 8 \end{bmatrix}$

3. Given that matrix $F = \begin{bmatrix} -8 & 14 & 2 \\ 7 & 3 & -5 \end{bmatrix}$, determine the order of matrix F. Hence, identify the elements f_{13}, f_{22} and f_{11}.

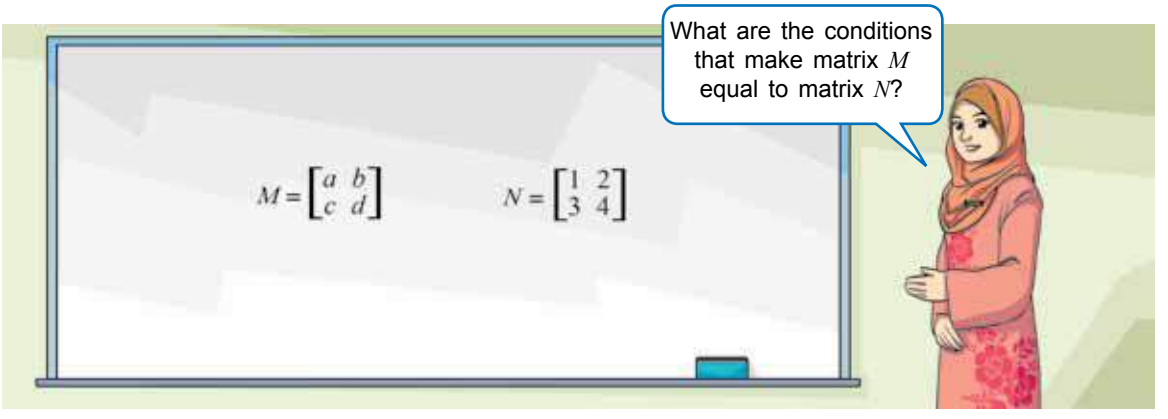
4. Given that matrix $B = \begin{bmatrix} 1 & -16 \\ 20 & 4 \end{bmatrix}$, calculate the value of $b_{12} + b_{21}$.

How to determine whether two matrices are equal?

Learning Standard

Determine whether two matrices are equal.

Look at the two matrices below.



It is found that matrix M and matrix N have the same order, which is 2×2 and if each corresponding element is equal, which is $a = 1$, $b = 2$, $c = 3$ and $d = 4$, then matrix M and matrix N are **equal** and can be written as $M = N$.

> $M = N$ if and only if both the matrices have the same order and each corresponding element is equal.

Example 4

Determine whether each of the following pairs of matrices is equal. Give your reason.

(a) $A = \begin{bmatrix} 2 & 11 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 11 \\ 1 & 3 \end{bmatrix}$

(b) $C = [3 \quad 9]$ and $D = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

(c) $E = \begin{bmatrix} 8 & 3 \\ -7 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 8 & -7 \\ 3 & 0 \end{bmatrix}$

(d) $G = \begin{bmatrix} 0.5 & -2 \\ 6 & 0.8 \\ -1 & 12 \end{bmatrix}$ and $H = \begin{bmatrix} \frac{1}{2} & -2 \\ 6 & \frac{4}{5} \\ -1 & 12 \end{bmatrix}$

Solution:

(a) $A = B$ because both the matrices have the same order and each corresponding element is equal.

(b) $C \neq D$ because both the matrices do not have the same order. The order of C is 1×2 while the order of D is 2×1.

(c) $E \neq F$ because the corresponding elements are different.

(d) $G = H$ because both the matrices have the same order and each corresponding element is equal.

Example 5

It is given that matrix $P = \begin{bmatrix} x & 7 \\ 0 & 5 - 3z \end{bmatrix}$ and matrix $Q = \begin{bmatrix} 5 & y + 1 \\ 0 & 2z \end{bmatrix}$. Determine the values of x, y and z if $P = Q$.

Solution:

$P = Q$, hence all the corresponding elements are equal.

$x = 5$, $7 = y + 1$, $5 - 3z = 2z$

$y = 7 - 1$ $5 = 5z$

$y = 6$ $z = 1$

Self Practice 2.1c

1. Determine whether each of the following pairs of matrices is equal.

(a) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 0.1 & 6 \\ -1 & 1.5 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{10} & 6 \\ -1 & \frac{3}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 12 \\ -7 \end{bmatrix}$ and $[12 \quad -7]$

(d) $\begin{bmatrix} 0 & 9 \\ 8 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 8 \\ 9 & 1 \end{bmatrix}$

2. Given that $P = Q$, calculate the values of x, y and z.

(a) $P = \begin{bmatrix} 6 & 0 \\ 3 & y \\ 2z - 3 & -5 \end{bmatrix}$ and $Q = \begin{bmatrix} x & 0 \\ 3 & 2 \\ -2 & -5 \end{bmatrix}$

(b) $P = \begin{bmatrix} 10 & -1 \\ 6y + 5 & 3z + 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 5x & -1 \\ 2y - 9 & -4x \end{bmatrix}$

2.2 Basic Operation on Matrices

How to add and subtract matrices?

Learning Standard

Add and subtract matrices.

How do you add or subtract these data?

	B	C	D	E	F	G	H	I	J	K	L
1	The sales of the fans in March				The sales of the fans in April				The sales of the fans in May		
2		Fan				Fan				Fan	
3		Stand	Ceiling	Wall	Stand	Ceiling	Wall		Stand	Ceiling	Wall
4	In-store	16	18	11	20	15	9		15	21	10
5	Online	5	10	4	7	12	5		10	24	10
6											
7		The total sales in March and April			The difference in sales between April and May						
8		Fan			Fan						
9		Stand	Ceiling	Wall	Stand	Ceiling	Wall				
10	In-store	36	33	20	-5	6	1				
11	Online	12	22	9	3	12	5				
12											

Observe the spreadsheet above that shows the sales of fans in Kedai Elektrik Sinar Jaya. The sales of fans in the months of March, April and May can be represented with matrix $P = \begin{bmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix}$, matrix $Q = \begin{bmatrix} 20 & 15 & 9 \\ 7 & 12 & 5 \end{bmatrix}$ and matrix $R = \begin{bmatrix} 15 & 21 & 10 \\ 10 & 24 & 10 \end{bmatrix}$ respectively.

The total sales in March and April can be obtained by adding the matrix P and matrix Q, that is

$$\begin{bmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix} + \begin{bmatrix} 20 & 15 & 9 \\ 7 & 12 & 5 \end{bmatrix} = \begin{bmatrix} 36 & 33 & 20 \\ 12 & 22 & 9 \end{bmatrix}.$$

The difference in sales between April and May can also be determined by performing the subtraction of matrix R and matrix Q, that is

$$\begin{bmatrix} 15 & 21 & 10 \\ 10 & 24 & 10 \end{bmatrix} - \begin{bmatrix} 20 & 15 & 9 \\ 7 & 12 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 6 & 1 \\ 3 & 12 & 5 \end{bmatrix}.$$

Addition and subtraction of matrices can only be performed on the matrices with the **same order**.

Each **corresponding element** is added or subtracted to obtain a single matrix with the same order.

For matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and matrix $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \text{ and } A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}.$$

Example 6

Determine whether addition and subtraction can be performed on the following pairs of matrices. Give your reason.

(a) $A = \begin{bmatrix} 2 & -5 & 3 \\ 8 & 11 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 6 \\ 4 & -1 \end{bmatrix}$

(b) $C = [1 \quad 12]$ and $D = [0 \quad -4]$

(c) $E = \begin{bmatrix} 15 & -4 \\ -1 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 2 & -5 \\ 13 & 0 \end{bmatrix}$

(d) $G = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $H = \begin{bmatrix} 8 & 7 \\ 16 & 1 \end{bmatrix}$

Solution:

(a) No because matrix A and matrix B have different orders.
(b) Yes because matrix C and matrix D have the same order.
(c) Yes because matrix E and matrix F have the same order.
(d) No because matrix G and matrix H have different orders.

Example 7

It is given that matrix $C = \begin{bmatrix} 10 & -8 & 4 \\ 6 & -11 & 7 \end{bmatrix}$, matrix $D = \begin{bmatrix} 14 & -2 & 1 \\ -3 & 5 & 9 \end{bmatrix}$, matrix $P = \begin{bmatrix} -2 & \frac{5}{6} & 6 \\ 7.4 & -13 & 5 \\ 1 & 9 & \frac{1}{4} \end{bmatrix}$ and matrix $Q = \begin{bmatrix} 18 & \frac{1}{3} & -7 \\ 2.5 & -8 & 3 \\ 12 & 0 & 0.4 \end{bmatrix}$. Calculate

(a) $C + D$

(b) $P - Q$

Solution:

(a) $C + D = \begin{bmatrix} 10 & -8 & 4 \\ 6 & -11 & 7 \end{bmatrix} + \begin{bmatrix} 14 & -2 & 1 \\ -3 & 5 & 9 \end{bmatrix}$

$= \begin{bmatrix} 10 + 14 & -8 + (-2) & 4 + 1 \\ 6 + (-3) & -11 + 5 & 7 + 9 \end{bmatrix}$ ← Add corresponding elements

$= \begin{bmatrix} 24 & -10 & 5 \\ 3 & -6 & 16 \end{bmatrix}$

(b) $P - Q = \begin{bmatrix} -2 & \frac{5}{6} & 6 \\ 7.4 & -13 & 5 \\ 1 & 9 & \frac{1}{4} \end{bmatrix} - \begin{bmatrix} 18 & \frac{1}{3} & -7 \\ 2.5 & -8 & 3 \\ 12 & 0 & 0.4 \end{bmatrix}$

$= \begin{bmatrix} -2 - 18 & \frac{5}{6} - \frac{1}{3} & 6 - (-7) \\ 7.4 - 2.5 & -13 - (-8) & 5 - 3 \\ 1 - 12 & 9 - 0 & \frac{1}{4} - 0.4 \end{bmatrix}$ ← Subtract corresponding elements

$= \begin{bmatrix} -20 & \frac{1}{2} & 13 \\ 4.9 & -5 & 2 \\ -11 & 9 & -0.15 \end{bmatrix}$

i-Technology

A scientific calculator can be used to perform the addition and subtraction of matrices. Scan the QR code or visit bit.do/VideoE201 to watch the related video.

Example 8

Given that matrix $D = \begin{bmatrix} 2x - 1 & -3 \\ -12 & 5 + y \end{bmatrix}$, matrix $E = \begin{bmatrix} x & 2 \\ 7 & y \end{bmatrix}$ and $D + E = \begin{bmatrix} 8 & -1 \\ -5 & 13 \end{bmatrix}$, calculate the values of x and y.

Info Bulletin

The matrices that are expressed in the form of equation is known as matrix equation.
For example, $A + B = C$.

Solution:

$$D + E = \begin{bmatrix} 8 & -1 \\ -5 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 2x - 1 & -3 \\ -12 & 5 + y \end{bmatrix} + \begin{bmatrix} x & 2 \\ 7 & y \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ -5 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 2x - 1 + x & -3 + 2 \\ -12 + 7 & 5 + y + y \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ -5 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 3x - 1 & -1 \\ -5 & 5 + 2y \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ -5 & 13 \end{bmatrix}$$ ← Equal matrices

$3x - 1 = 8$ and $5 + 2y = 13$ ← Compare the corresponding elements

$3x = 9$ $\quad$ $2y = 8$

$x = 3$ $\quad$ $y = 4$

Hence, $x = 3$ and $y = 4$

Example 9

Given that $F + \begin{bmatrix} 16 \\ -3 \end{bmatrix} - \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, calculate matrix F.

Solution:

$$F + \begin{bmatrix} 16 \\ -3 \end{bmatrix} - \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$F = \begin{bmatrix} -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 16 \\ -3 \end{bmatrix} + \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -11 \\ 16 \end{bmatrix}$$

Checking Answer

Substitute matrix $F = \begin{bmatrix} -11 \\ 16 \end{bmatrix}$ into the equation.

$$\begin{bmatrix} -11 \\ 16 \end{bmatrix} + \begin{bmatrix} 16 \\ -3 \end{bmatrix} - \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -11 + 16 - 7 \\ 16 + (-3) - 10 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Example 10

The table on the right shows the stock records of Form 4 textbooks of Science, Mathematics and Economy in SMK Taman Suria. Calculate the closing stock of each type of textbook.

	Science	Mathematics	Economy
Opening stock	326	335	82
New books received	56	47	15
Lost and damaged books	32	26	11

Solution:

Closing stock = Opening stock + New books received – Lost and damaged books

= [326 335 82] + [56 47 15] – [32 26 11]

= [350 356 86]

Hence, the closing stock of Science, Mathematics and Economy textbooks are 350, 356 and 86 respectively.

Self Practice 2.2a

1. Determine whether addition and subtraction can be performed on each of the following pairs of matrices.

(a) $\begin{bmatrix} -5 & 9 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 7 \\ 6 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 13 & -1 & 11 \\ -2 & 8 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & -16 & 7 \\ 1 & 5 & 0 \\ 3 & 2 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 12 & -7 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -9 \end{bmatrix}$ and $\begin{bmatrix} 1 & 6 \end{bmatrix}$

2. Given that matrix $P = \begin{bmatrix} 12 & 1 \\ -3 & 4 \end{bmatrix}$, matrix $Q = \begin{bmatrix} 8 & -2 \\ 0 & 5 \end{bmatrix}$ and matrix $R = \begin{bmatrix} 6 & 3 \\ 7 & -1 \end{bmatrix}$, calculate

(a) $P - Q + R$

(b) $P + Q - R$

3. Solve each of the following.

(a) $\begin{bmatrix} 12 & 10 & 1 \\ -4 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 9 \\ 2 & 8 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 18 & -3 \\ -7 & 15 \end{bmatrix} - \begin{bmatrix} 11 & 5 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ 9 \end{bmatrix} - \begin{bmatrix} 19 \\ -3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 8 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 14 & 6 \\ -1 & 15 \end{bmatrix}$

4. Given that $\begin{bmatrix} 3a + 2 \\ 9 - b \end{bmatrix} + \begin{bmatrix} 4 \\ 2b \end{bmatrix} = \begin{bmatrix} 15 \\ -8 \end{bmatrix}$, calculate the values of a and b.

5. Given that matrix $S = \begin{bmatrix} 4x + 1 & -5 \\ 6 - y & x \end{bmatrix}$, matrix $T = \begin{bmatrix} x & 6 \\ 7 & 3y \end{bmatrix}$ and $S - T = \begin{bmatrix} 10 & -11 \\ -2 & z \end{bmatrix}$, calculate the values of x, y and z.

6. Given that $\begin{bmatrix} 3 & -4 \\ 1 & 0 \\ -6 & 7 \end{bmatrix} + \begin{bmatrix} -7 & 2 \\ 9 & 6 \\ 10 & 8 \end{bmatrix} - V = \begin{bmatrix} 11 & -4 \\ -1 & 5 \\ 6 & 9 \end{bmatrix}$, calculate matrix V.

7. Mr Gopal has two stores, A and B. The tables below show the income and expenses for the sales of food and drink in both stores for the month of June.

Income		
	Food	Drink
Store A	RM2 650	RM1 890
Store B	RM1 560	RM910

Expenses		
	Food	Drink
Store A	RM930	RM850
Store B	RM540	RM260

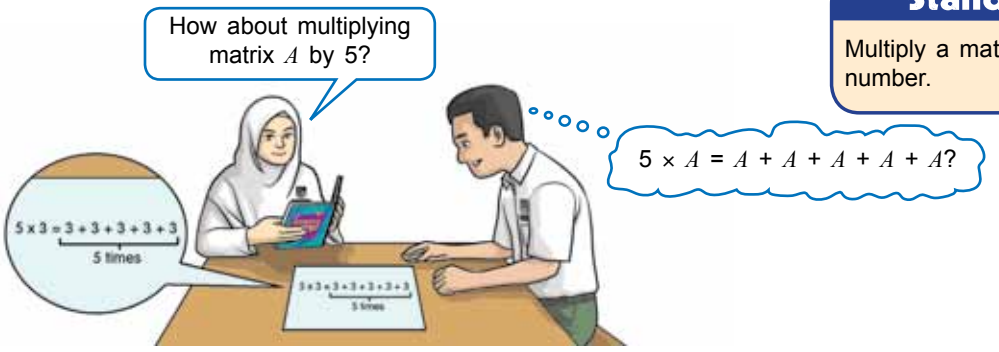
Calculate the total profit earned by Mr Gopal from each of his stores in the month of June. Show your calculation in the form of matrices.

[It is given that profit = income – expense]

How to multiply a matrix by a number?

Learning Standard

Multiply a matrix by a number.



Multiplication of a matrix by a number is a process of **repeated addition** of the matrix. If matrix A is multiplied by a number n, then matrix A can be added to the same matrix A repeatedly for n times, that is

$$nA = \underbrace{A + A + \ldots + A}_{n \text{ times}}.$$

This formula means that each element in matrix A is added to the same element repeatedly for n times. Therefore, to multiply a matrix by a number, multiply every element in the matrix with the number.

It is given that matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and n is a number.

Hence, $nA = n\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} na & nb \\ nc & nd \end{bmatrix}$.

n is known as a scalar.

Info Bulletin

When a matrix is multiplied by a real number, that real number is called a scalar.

Multiplication of a matrix by a number is known as **scalar multiplication**.

Example 11

Given that $D = \begin{bmatrix} -5 & 4 \\ 2 & 1 \end{bmatrix}$, calculate

(a) $3D$

(b) $-\frac{1}{2}D$

Solution:

(a) $3D = 3\begin{bmatrix} -5 & 4 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3(-5) & 3(4) \\ 3(2) & 3(1) \end{bmatrix}$ ← Multiply all elements by 3

$= \begin{bmatrix} -15 & 12 \\ 6 & 3 \end{bmatrix}$

(b) $-\frac{1}{2}D = -\frac{1}{2}\begin{bmatrix} -5 & 4 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} \left(-\frac{1}{2}\right)(-5) & \left(-\frac{1}{2}\right)(4) \\ \left(-\frac{1}{2}\right)(2) & \left(-\frac{1}{2}\right)(1) \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{2} & -2 \\ -1 & -\frac{1}{2} \end{bmatrix}$

MIND MOBILISATION 1 Group

Aim: To explore the laws of arithmetic operations in the addition and subtraction of matrices.

Steps:

1. Divide the class into groups of 4 pupils.
2. Determine the result of the addition and subtraction in the **Activity Sheet** below.

Activity Sheet:

$A = \begin{bmatrix} 2 & 7 \\ 6 & 11 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 3 \\ 5 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 9 & 2 \\ 10 & -1 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(a)

Commutative Law	
$A + B$	$B + A$
$A - B$	$B - A$

(b)

Distributive Law	
$h(A + B)$	$hA + hB$
$h(A - B)$	$hA - hB$

(c)

Associative Law	
$(A + B) + C$	$A + (B + C)$
$(A - B) - C$	$A - (B - C)$

(d)

Addition and Subtraction of Zero Matrix	
$A + O$	$A - O$

Discussion:

Based on the results obtained in each of the tables above, what conclusion can be made? What is the relation between the process of addition and subtraction of matrices with the laws of arithmetic operations?

The results of Mind Mobilisation 1 show that;

(a) $A + B = B + A$. Addition of matrices obeys the **Commutative Law**.
$A - B \neq B - A$. Subtraction of matrices does not obey the Commutative Law.

(b) $h(A + B) = hA + hB$, $h(A - B) = hA - hB$.
Addition and subtraction of matrices obey the **Distributive Law**.

(c) $(A + B) + C = A + (B + C)$. Addition of matrices obeys the **Associative Law**.
$(A - B) - C \neq A - (B - C)$. Subtraction of matrices does not obey the Associative Law.

(d) A matrix with all its elements equal to zero is called **zero matrix**, for example $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Addition and subtraction of matrix A with zero matrix, O are:
$A + O = A$ and $A - O = A$

Info Bulletin

Example of zero matrix:
$O_{1 \times 2} = [0 \quad 0]$
$O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Example 12

Given that $P = \begin{bmatrix} 7 & 9 \\ -3 & 8 \\ 6 & 12 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & -3 \\ 1 & 5 \\ 0 & 4 \end{bmatrix}$, calculate $3(P - Q)$.

Solution:

$$3(P - Q) = 3\left(\begin{bmatrix} 7 & 9 \\ -3 & 8 \\ 6 & 12 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 1 & 5 \\ 0 & 4 \end{bmatrix}\right)$$

Perform subtraction inside the bracket

$$= 3\begin{bmatrix} 5 & 12 \\ -4 & 3 \\ 6 & 8 \end{bmatrix}$$

Multiply all the elements by 3

$$= \begin{bmatrix} 15 & 36 \\ -12 & 9 \\ 18 & 24 \end{bmatrix}$$

Interactive Platform

Show that $3P - 3Q = 3(P - Q)$

i-Technology

A scientific calculator can be used to perform the scalar multiplication of matrices. Scan the QR code or visit bit.do/VideoE202 to watch the related video.

Example 13

(a) Given that $\frac{1}{2}\begin{bmatrix} 4 \\ 12 \end{bmatrix} - \begin{bmatrix} x \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ y \end{bmatrix}$, calculate the values of x and y.

(b) Given that $4R + \begin{bmatrix} 9 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 10 & 1 \end{bmatrix}$, calculate matrix R.

Solution:

(a) $\frac{1}{2}\begin{bmatrix} 4 \\ 12 \end{bmatrix} - \begin{bmatrix} x \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ y \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} x \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2 - x \\ 6 - (-3) \end{bmatrix} = \begin{bmatrix} 5 \\ y \end{bmatrix}$$

Compare the corresponding elements.

$2 - x = 5$, $6 - (-3) = y$

$x = -3$ $\quad$ $y = 9$

(b) $4R + \begin{bmatrix} 9 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 10 & 1 \end{bmatrix}$

$$4R = \begin{bmatrix} -3 & 4 \\ 10 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 2 & -3 \end{bmatrix}$$

$$4R = \begin{bmatrix} -12 & 4 \\ 8 & 4 \end{bmatrix}$$

$4R = A$, $R = \frac{A}{4} = \frac{1}{4}A$

$$R = \frac{1}{4}\begin{bmatrix} -12 & 4 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$$

Example 14

The average number of vehicles per day in a parking area for 5 working days is represented in Table X. Table Y represents the average number of vehicles per day on weekends.

	Car	Motorcycle
Covered	42	8
Non-covered	20	11

Table X

	Car	Motorcycle
Covered	25	5
Non-covered	12	3

Table Y

Calculate the number of vehicles parked at the parking area in a week.

Solution:

$$5X + 2Y = 5\begin{bmatrix} 42 & 8 \\ 20 & 11 \end{bmatrix} + 2\begin{bmatrix} 25 & 5 \\ 12 & 3 \end{bmatrix}$$

← 5 working days + 2 days during the weekend

$$= \begin{bmatrix} 210 & 40 \\ 100 & 55 \end{bmatrix} + \begin{bmatrix} 50 & 10 \\ 24 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 260 & 50 \\ 124 & 61 \end{bmatrix}$$

Self Practice 2.2b

1. Determine the product of each of the following matrices.

(a) $3\begin{bmatrix} -7 \\ 2 \end{bmatrix}$ (b) $0.6[11 \quad 5]$ (c) $\frac{1}{4}\begin{bmatrix} 12 & -20 \\ -6 & 16 \\ 9 & 1 \end{bmatrix}$

(d) $-2\begin{bmatrix} 0.4 & 8 \\ -9 & 2.5 \end{bmatrix}$ (e) $1.2\begin{bmatrix} 10 & -1 & 11 \\ 3 & 7 & -5 \end{bmatrix}$ (f) $-\frac{1}{20}\begin{bmatrix} 100 \\ -90 \\ -20 \end{bmatrix}$

2. Solve each of the following operations.

(a) $5\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 6 & 1 \\ -1 & 8 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 10 & 2 \\ 9 & -4 \\ -6 & 14 \end{bmatrix}$ (b) $6\begin{bmatrix} -1 \\ 4 \end{bmatrix} - 0.5\begin{bmatrix} 8 \\ 14 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

(c) $7[3 \quad -2 \quad 1] - \frac{1}{3}[21 \quad 6 \quad -9]$ (d) $0.2\begin{bmatrix} 10 & -25 \\ -6 & 8 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 15 & 20 \\ -5 & 2.5 \end{bmatrix}$

3. Given that matrix $E = \begin{bmatrix} 9 & 6 \\ 2 & 11 \end{bmatrix}$, matrix $F = \begin{bmatrix} -7 & 22 \\ 3 & 4 \end{bmatrix}$ and matrix $G = \begin{bmatrix} -1 & 10 \\ -8 & 5 \end{bmatrix}$, show that $(E + F) + G = E + (F + G)$.

4. Given that matrix $P = \begin{bmatrix} \frac{1}{2} \\ -0.7 \end{bmatrix}$, matrix $Q = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ and matrix $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, calculate $P - 1.4Q + O$.

5. Given that $4\begin{bmatrix} 2 & 3 \\ -5 & b \end{bmatrix} - \begin{bmatrix} -4 & 9 \\ c & 0.1 \end{bmatrix} = \frac{3}{5}\begin{bmatrix} a & 5 \\ -35 & 1.5 \end{bmatrix}$, calculate the values of a, b and c.

6. Given that $[-10 \quad 9] - 2X + 5[2 \quad 1] = [3 \quad 8]$, calculate matrix X.

7. The shoe shop owned by Encik Jamal sells adult and children shoes. Table 1 shows the stock for each type of shoes at the beginning of a week while Table 2 shows the sales of each type of shoes of that week.

	Female	Male
Adult	85	70
Children	110	98

Table 1

	Female	Male
Adult	33	24
Children	42	40

Table 2

Calculate the closing inventory of each type of shoes at the end of the week. Show your calculation in the form of matrices.

How to multiply two matrices?

Learning Standard

Multiply two matrices.

Based on the previous situation, the sales of fans in the month of March can be represented with a matrix as follows:

$$P = \begin{matrix} \\ \text{In-store} \\ \text{Online} \end{matrix} \begin{matrix} \begin{bmatrix} \text{Stand} & \text{Ceiling} & \text{Wall} \\ 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix} \end{matrix}$$

The commissions for selling each unit of the stand fans, ceiling fans and wall fans are RM25, RM30 and RM20 respectively. How do you calculate the total commission earned from the in-store and online sales?

Total commission earned from the in-store sales	Total commission earned from the online sales
$= (16 \times \text{RM25}) + (18 \times \text{RM30}) + (11 \times \text{RM20})$	$= (5 \times \text{RM25}) + (10 \times \text{RM30}) + (4 \times \text{RM20})$
$= \text{RM1 160}$	$= \text{RM505}$

The total commission earned can be calculated in the form of a matrix. If the commission for selling each unit of the fans is represented in the form of column matrix, that is $K = \begin{bmatrix} 25 \\ 30 \\ 20 \end{bmatrix}$, then the total commission earned from the in-store and online sales can be written in the form of matrix as follows.

$$PK = \begin{bmatrix} 16 & 18 & 11 \\ 5 & 10 & 4 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 25 \\ 30 \\ 20 \end{bmatrix}_{3 \times 1}$$

$$= \begin{bmatrix} 16(25) + 18(30) + 11(20) \\ 5(25) + 10(30) + 4(20) \end{bmatrix}$$

$$= \begin{bmatrix} 1160 \\ 505 \end{bmatrix}_{2 \times 1}$$

Critical Mind

Assume K is a row matrix. Perform the multiplication on matrices, PK. Is the multiplication possible?

PK is known as the multiplication of matrix P with matrix K and $\begin{bmatrix} 1160 \\ 505 \end{bmatrix}$ is the product of the two matrices.

In general, to multiply two matrices, A and B, the number of columns in matrix A must be the same with the number of rows in matrix B. The number of rows in matrix A and the number of columns in matrix B become the order of the product of the two matrices, AB.

Order : $A_{m \times n}$ $B_{n \times p}$ $= AB_{m \times p}$

Number of columns in A = Number of rows in B

The order of AB is $m \times p$

If matrix A has an order of $m \times n$ and matrix B has an order of $n \times p$, then the multiplication AB can be performed and the order of AB is $m \times p$.

Example 15

It is given that matrix $C = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ and matrix $D = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}$. Determine whether multiplication CD and DC can be performed. If yes, state the order of the product of the matrices.

Solution:

Order of matrix $C = 2 \times 1$, order of matrix $D = 2 \times 2$

C $\quad$ D

2×1 $\quad$ 2×2

Not the same

D $\quad$ C

2×2 $\quad$ 2×1

Same

Order of DC

Multiplication CD cannot be performed because the number of columns in matrix C is not the same with the number of rows in matrix D.

Multiplication DC can be performed because the number of columns in matrix D is the same with the number of rows in matrix C. The order of DC is 2×1.

Example 16

It is given that matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix}$. Calculate AB.

Solution:

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} (2)(6) + (3)(-2) & \\ & \end{bmatrix} = \begin{bmatrix} 6 & \\ & \end{bmatrix}$$

The element at row 1 and column 1

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & (2)(-7) + (3)(1) \\ & \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ & \end{bmatrix}$$

The element at row 1 and column 2

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ (1)(6) + (5)(-2) & \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -4 & \end{bmatrix}$$

The element at row 2 and column 1

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -4 & (1)(-7) + (5)(1) \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -4 & -2 \end{bmatrix}$$

The element at row 2 and column 2

Hence, $AB = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 6 & -7 \\ -2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (2)(6) + (3)(-2) & (2)(-7) + (3)(1) \\ (1)(6) + (5)(-2) & (1)(-7) + (5)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -11 \\ -4 & -2 \end{bmatrix}$$

Smart TiPS

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}_{2 \times 1}$$

$$= \begin{bmatrix} (a_{11} \times b_{11}) + (a_{12} \times b_{21}) \\ (a_{21} \times b_{11}) + (a_{22} \times b_{21}) \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} + d_{11} \\ e_{21} + f_{21} \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} \\ h_{21} \end{bmatrix}_{2 \times 1}$$

Example 17

It is given that matrix $E = \begin{bmatrix} 5 & -1 & 0 \\ -4 & 8 & 7 \end{bmatrix}$, matrix $F = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$ and matrix $G = [4 \quad 3]$. Calculate

(a) GE (b) FG (c) GF

Solution:

(a) $GE = [4 \quad 3]_{1 \times 2} \begin{bmatrix} 5 & -1 & 0 \\ -4 & 8 & 7 \end{bmatrix}_{2 \times 3}$

$= [4(5) + 3(-4) \quad 4(-1) + 3(8) \quad 4(0) + 3(7)]$

$= [8 \quad 20 \quad 21]_{1 \times 3}$ ← The product is a matrix with order 1×3

(b) $FG = \begin{bmatrix} -2 \\ 7 \end{bmatrix}_{2 \times 1} [4 \quad 3]_{1 \times 2}$

$= \begin{bmatrix} (-2)(4) & (-2)(3) \\ 7(4) & 7(3) \end{bmatrix}$

$= \begin{bmatrix} -8 & -6 \\ 28 & 21 \end{bmatrix}_{2 \times 2}$ ← The product is a matrix with order 2×2

(c) $GF = [4 \quad 3]_{1 \times 2} \begin{bmatrix} -2 \\ 7 \end{bmatrix}_{2 \times 1}$

$= [4(-2) + 3(7)]$

$= [13]_{1 \times 1}$ ← The product is a matrix with order 1×1

Example 18

It is given that matrix $K = \begin{bmatrix} 3 & 2 \\ -1 & 5 \\ 4 & -2 \end{bmatrix}$ and matrix $L = \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix}$. Calculate

(a) L^2 (b) KL^2 (c) L^3

Solution:

(a) $L^2 = LL$

$= \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 16 + 0 & -8 + 2 \\ 0 + 0 & 0 + 1 \end{bmatrix}$

$= \begin{bmatrix} 16 & -6 \\ 0 & 1 \end{bmatrix}$

Smart TIPS

Always check the order of the matrix before performing multiplication. For example, when a matrix with order $m \times n$ is multiplied with a matrix with order $n \times p$, it produces a matrix with order $m \times p$.

Critical Mind

Why $FG \neq GF$?

i-Technology

A scientific calculator can be used to perform the multiplication of two matrices. Scan the QR code or visit bit.do/VideoE203 to watch the related video.

Info Bulletin

Given that $L = \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix}$,

$L^2 \neq \begin{bmatrix} (-4)^2 & 2^2 \\ 0^2 & 1^2 \end{bmatrix}$.

$L^2 = LL$,

$L^3 = L^2L = LL^2$.

(b) KL^2

$$= \begin{bmatrix} 3 & 2 \\ -1 & 5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 16 & -6 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 48 + 0 & -18 + 2 \\ -16 + 0 & 6 + 5 \\ 64 + 0 & -24 + (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 48 & -16 \\ -16 & 11 \\ 64 & -26 \end{bmatrix}$$

(c) L^3

$$= L^2L$$

$$= \begin{bmatrix} 16 & -6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -64 + 0 & 32 + (-6) \\ 0 + 0 & 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} -64 & 26 \\ 0 & 1 \end{bmatrix}$$

Critical Mind

Try to calculate LL^2.

Mathematics is fun!

Scan the QR code or visit bit.do/MatrixCalculator to use a matrix calculator.

Example 19

Given that $[6x \quad -5] \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = [7 \quad 1 - y]$, calculate the values of x and y.

Solution:

$$[6x \quad -5] \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = [7 \quad 1 - y]$$

$$[6x \quad -5] \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = [(6x)(2) + (-5)(1) \quad (6x)(3) + (-5)(4)]_{1 \times 2}$$

$$= [12x - 5 \quad 18x - 20]$$

Hence, $[12x - 5 \quad 18x - 20] = [7 \quad 1 - y]$.

Compare the corresponding elements.

$12x - 5 = 7$	and	$18x - 20 = 1 - y$
$12x = 12$		$18(1) - 20 = 1 - y$
$x = 1$		$-2 = 1 - y$
		$y = 3$

Critical Mind

It is given that matrix $P = [a \quad b + 1]$ and matrix $Q = \begin{bmatrix} c - 1 & 2 \\ -3 & 4d \end{bmatrix}$.

Calculate

(i) PQ

(ii) Q^2

Example 20

The table below shows the share units purchased by Khairil and Mahmud.

	Share A (unit)	Share B (unit)
Khairil	5 000	4 000
Mahmud	2 000	6 000

It is given that the prices of 1 unit of share A and 1 unit of share B during the purchase are RM1.50 and RM0.82 respectively. Calculate the total investment of Khairil and the total investment of Mahmud.

Solution:

$$\begin{bmatrix} 5000 & 4000 \\ 2000 & 6000 \end{bmatrix} \begin{bmatrix} 1.50 \\ 0.82 \end{bmatrix} = \begin{bmatrix} 7500 + 3280 \\ 3000 + 4920 \end{bmatrix}$$

$$= \begin{bmatrix} 10780 \\ 7920 \end{bmatrix}$$

The total investments of Khairil and Mahmud are RM10 780 and RM7 920 respectively.

Self Practice 2.2c

1. Given that four matrices, $P = \begin{bmatrix} 3 & 6 \\ -1 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $R = [4 \quad 8 \quad 5]$ and $S = \begin{bmatrix} 0 & -6 & 1 \\ 3 & 11 & -2 \end{bmatrix}$, determine whether the following multiplication of matrices can be performed. If yes, state the order of the product of the pairs of matrices.

(a) PQ (b) QR (c) RS
(d) SP (e) PS (f) QP

2. It is given that four matrices, $T = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 2 & -1 \end{bmatrix}$, $U = \begin{bmatrix} 0 & -4 \\ -3 & 5 \\ 1 & 2 \end{bmatrix}$, $V = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$ and $W = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$. Calculate

(a) TU (b) UW (c) UV
(d) WV (e) W^2 (f) W^3

3. Given that $\begin{bmatrix} -1 & x \\ y & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 31 \\ 29 \end{bmatrix}$, calculate the values of x and y.

4. Given that $\begin{bmatrix} 9 & r \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 4 & s \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 14.5 \\ 8 & 8 \end{bmatrix}$, calculate the values of r and s.

5. Given that $G = \begin{bmatrix} p & 5 \\ 1 & -4 \end{bmatrix}$ and $H = \begin{bmatrix} -6 & 7 \\ 3 & 0 \end{bmatrix}$, calculate the values of p, q and r if

(a) $GH = \begin{bmatrix} 3 & 2q \\ -18 & 3p + r \end{bmatrix}$ (b) $G^2 = \begin{bmatrix} r & -25 \\ -5 & 7q \end{bmatrix}$

(c) $HG = \begin{bmatrix} -11 & 2.5q \\ \frac{p + 3r}{2} & 5p \end{bmatrix}$ (d) $H^2 = \begin{bmatrix} 57 & 6p \\ 1.2q & \frac{7r}{5} \end{bmatrix}$

6. Mr Koh rents a booth in Educational Expo to sell three types of goods as shown in the table below.

	Goods A	Goods B	Goods C
First day	40	28	36
Second day	42	36	30
Third day	35	25	42

It is given that the profits of one item sold for selling goods A, B and C are RM5, RM8 and RM6 respectively. Calculate the total profit earned by Mr Koh every day. Show your calculation in the form of matrices.

[It is given that total profit = sales of goods A × profit of goods A
+ sales of goods B × profit of goods B
+ sales of goods C × profit of goods C]

What are the characteristics of identity matrix?

$$a \times 1 = a$$
$$1 \times a = a$$

Learning Standard

Explain the characteristics of identity matrix.

When 1 is multiplied with any number, a, its product is a. When a matrix is multiplied with matrix A, its product is matrix A. Then the first matrix is the identity matrix. What are the characteristics of an identity matrix?

MIND MOBILISATION 2 Group

Aim: To determine the identity matrix.

Steps:

1. Divide the class into groups of 4 pupils.
2. Copy the **Activity Sheet** below and take turn to complete it.

Activity Sheet:

	Matrix A	Matrix B	AB	BA
(a)	$\begin{bmatrix} 5 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$		
(b)	$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$		
(c)	$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		
(d)	$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		
(e)	$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 1 \\ 5 & 3 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$		
(f)	$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 1 \\ 5 & 3 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$		
(g)	$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 1 \\ 5 & 3 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

Discussion:

1. Which matrix B when multiplied with matrix A will result in matrix A?
2. What are the elements in matrix B? How are the elements positioned in matrix B?

The results of Mind Mobilisation 2 show that;

(a) Matrix B in the form of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ when multiplied with matrix A will result in matrix A, that is $AB = BA = A$.

(b) Elements in the matrices B consist of 0 and 1 only with element 1 placed along the diagonal from the top left corner to the bottom right corner and the other elements are zero.

Matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is known as identity matrix and is denoted by I.

Identity matrices are square matrices. **Identity matrix**, I, when multiplied with a matrix A, will result in matrix A.

$$AI = IA = A$$

Smart TIPS

Identity matrix, I with order $n \times n$:

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Info Bulletin

Arrange element 1 along the diagonal from the top left corner to the bottom right corner and the other elements are zero. The diagonal is called main diagonal. This matrix is called a diagonal matrix.

Example 21

Write the identity matrix based on the order given below.

(a) 1×1 (b) 2×2

(c) 4×4 (d) 5×5

Solution:

(a) $[1]$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Critical Mind

Identity matrix is a diagonal matrix. Is the diagonal matrix an identity matrix? Discuss.

Self Practice 2.2d

1. Which of the following matrices are identity matrix? If it is not an identity matrix, give your reason.

(a) $[0 \quad 1]$

(b) $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

2. It is given that matrix $C = \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix}$ and matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that matrix D is an identity matrix.

3. It is given that matrix $S = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$ and matrix $T = \begin{bmatrix} 3 & 1 \\ -5 & 4 \end{bmatrix}$. Calculate

(a) $SI + TI$ (b) $(IS)T$ (c) $4IT - I^2$ (d) $(S - I)I$

What is the meaning of inverse matrix?

Learning Standard

Explain the meaning of inverse matrix and hence determine the inverse matrix for a 2 × 2 matrix.

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = BA = I$$

If multiplication of matrix A and matrix B produces an identity matrix, I, then matrix B is the inverse of matrix A and vice versa.

MIND MOBILISATION 3 Group

Aim: To determine the inverse matrix.

Steps:

1. Divide the class into groups of 4 pupils.
2. Each pupil chooses a piece of matrix A card and a piece of matrix B card as shown below.

$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$	$A = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$	$A = \begin{bmatrix} 2 & 0 \\ 6 & -1 \end{bmatrix}$	$A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$
$B = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$	$B = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$	$B = \begin{bmatrix} -1 & 6 \\ 0 & 2 \end{bmatrix}$	$B = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

3. Pupils perform the multiplication AB and BA. The product is recorded in the table as shown on the right.
4. Pupils exchange matrix B card with others in the group. Step **3** is repeated.
5. Pupils discuss their results in groups.

Matrix A	Matrix B	AB	BA

Discussion:

1. Which two matrices result in identity matrix after performing multiplication?
2. What is the conclusion about the relationship between the two matrices?

The results of Mind Mobilisation 3 show that

$$\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{bmatrix}7 & -3\\-2 & 1\end{bmatrix} = \begin{bmatrix}7 & -3\\-2 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}4 & -3\\3 & -2\end{bmatrix}\begin{bmatrix}-2 & 3\\-3 & 4\end{bmatrix} = \begin{bmatrix}-2 & 3\\-3 & 4\end{bmatrix}\begin{bmatrix}4 & -3\\3 & -2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}-1 & 2\\2 & -3\end{bmatrix} = \begin{bmatrix}-1 & 2\\2 & -3\end{bmatrix}\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

The pairs of matrices above are inverse matrices of one another.

Info Bulletin

A^{-1} is read as "inverse matrix of A".

$$A^{-1} \neq \frac{1}{A}$$

Multiplication of matrix A and **inverse matrix** of A, A^{-1}, will result in identity matrix, I.

$$AA^{-1} = A^{-1}A = I$$

Example 22

Determine whether the following matrix is the inverse matrix of $\begin{bmatrix}4 & 1\\7 & 2\end{bmatrix}$. Explain your answer.

(a) $\begin{bmatrix}4 & 1\\-7 & 2\end{bmatrix}$ (b) $\begin{bmatrix}2 & -1\\-7 & 4\end{bmatrix}$

Solution:

(a) $\begin{bmatrix}4 & 1\\7 & 2\end{bmatrix}\begin{bmatrix}4 & 1\\-7 & 2\end{bmatrix} = \begin{bmatrix}9 & 6\\14 & 11\end{bmatrix}$ ← The product is not an identity matrix

$\begin{bmatrix}4 & 1\\-7 & 2\end{bmatrix}$ is not the inverse matrix of $\begin{bmatrix}4 & 1\\7 & 2\end{bmatrix}$ because the product of these two matrices is not an identity matrix.

(b) $\begin{bmatrix}4 & 1\\7 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\-7 & 4\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ ← The product is an identity matrix

$\begin{bmatrix}2 & -1\\-7 & 4\end{bmatrix}\begin{bmatrix}4 & 1\\7 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ ← The product is an identity matrix

$\begin{bmatrix}2 & -1\\-7 & 4\end{bmatrix}$ is the inverse matrix of $\begin{bmatrix}4 & 1\\7 & 2\end{bmatrix}$ because the product of these two matrices is an identity matrix.

Info Bulletin

Inverse matrix only exists in the form of square matrix because both AA^{-1} and $A^{-1}A$ equal to I. However, not all the square matrices have inverse matrices.

Self Practice 2.2e

1. Determine whether the following matrices are inverse matrices of one another.

(a) $\begin{bmatrix}5 & 4\\3 & 2\end{bmatrix}, \begin{bmatrix}2 & -4\\-3 & 5\end{bmatrix}$ (b) $\begin{bmatrix}1 & 3\\1 & 4\end{bmatrix}, \begin{bmatrix}\frac{1}{2} & 3\\1 & 2\end{bmatrix}$

(c) $\begin{bmatrix}1 & 2\\4 & 9\end{bmatrix}, \begin{bmatrix}1 & -2\\-4 & 9\end{bmatrix}$ (d) $\begin{bmatrix}-2 & 3\\-5 & 7\end{bmatrix}, \begin{bmatrix}7 & -3\\5 & -2\end{bmatrix}$

How to determine the inverse matrix for a 2 × 2 matrix?

MIND MOBILISATION 4 Group

Aim: To derive the formula used to determine an inverse matrix for 2 × 2 matrix.

Steps:

1. Divide the class into groups of 4 pupils.
2. Copy and complete the following **Activity Sheet** by following the given instructions.

Activity Sheet:

It is given that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

Instruction	Steps	
Multiply matrix A and A^{-1}	$AA^{-1} = \begin{bmatrix} ap + br & \square \\ \square & \square \end{bmatrix}$	
Form 4 equations from $AA^{-1} = I$	$\begin{bmatrix} ap + br & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (i) $ap + br = 1$ (ii) $\square$ (iii) $\square$ (iv) $\square$	
With substitution method, express p, q, r and s in terms of a, b, c and d	Use the equations in (i) and (iii), express p and r in terms of a, b, c and d.	Use the equations in (ii) and (iv), express q and s in terms of a, b, c and d.
	$p = \dfrac{d}{ad - bc}$ $r = \square$	$q = \square$ $s = \square$
Write matrix A^{-1} in terms of a, b, c and d	$A^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \square \\ \square & \square \end{bmatrix}$	
Write A^{-1} as scalar multiplication	$A^{-1} = \dfrac{1}{\square} \begin{bmatrix} d & \square \\ \square & \square \end{bmatrix}$	

3. Write all the steps on a piece of paper and paste it on the wall. Pupils give comments by attaching sticky notes on the findings of other groups.

Discussion:
What is the formula used to determine an inverse matrix?

The results of Mind Mobilisation 4 show that

(a) the scalar is $\frac{1}{ad - bc}$,

(b) the element a_{11} is d, element a_{12} is $-b$, element a_{21} is $-c$ and element a_{22} is a.
Note that the position of a and d are exchanged while b and c are multiplied by -1.

Given matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse matrix, A^{-1} can be obtained using the formula as follows:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ where } ad - bc \neq 0$$

Determinant for matrix A, $|A|$ is $ad - bc$ where $ad - bc \neq 0$. Therefore inverse matrix, A^{-1} exists. Inverse matrix does not exist when $ad - bc = 0$.

Historical Treasure

The term determinant is introduced in 1801 by Carl Gauss (1777-1855), a German mathematician.

Example 23

For each of the following matrices, determine whether the inverse matrix exists. If yes, calculate the inverse matrix.

(a) $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$

(b) $B = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

Solution:

(a) $ad - bc = 1(8) - 2(4)$
$= 8 - 8$
$= 0$

Determine the existence of inverse matrix with matrix determinant, $ad - bc$

$|A| = 0$. Hence, A^{-1} does not exist.

(b) $ad - bc = 3(4) - 5(2)$
$= 12 - 10$
$= 2$
$\neq 0$

$|B| \neq 0$. Hence, B^{-1} exists.

$$B^{-1} = \frac{1}{3(4) - 5(2)}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$$

Exchange the position of the elements in the main diagonal and multiply the other two elements with -1

Example 24

Given that matrix $C = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$, calculate the inverse matrix of C.

Solution:

$$C^{-1} = \frac{1}{2(-2) - (-6)(1)}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

i-Technology

A scientific calculator can be used to find the determinant of a matrix. Scan the QR code or visit bit.do/VideoE204 to watch the related video.

Example 25

Given that matrix $D = \begin{bmatrix} m & -6 \\ 1 & -2 \end{bmatrix}$, calculate the value of m if

(a) matrix D does not have inverse matrix,

(b) $D^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$.

Solution:

(a) Matrix D does not have inverse matrix, hence

$$\begin{aligned} ad - bc &= 0 \\ -2m - (-6)(1) &= 0 \\ -2m + 6 &= 0 \\ m &= 3 \end{aligned}$$

(b)
$$DD^{-1} = I$$
$$\begin{bmatrix} m & -6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$m(-1) + (-6)\left(-\frac{1}{2}\right) = 1$$
$$\begin{aligned} -m + 3 &= 1 \\ m &= 2 \end{aligned}$$

Critical Mind

What is the inverse matrix of identity matrix?

Critical Mind

Why does A^{-1} not exist when $ad - bc = 0$?

Checking Answer

$$D^{-1} = \frac{1}{-2m - (-6)(1)}\begin{bmatrix} -2 & 6 \\ -1 & m \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} = \frac{1}{-2m + 6}\begin{bmatrix} -2 & 6 \\ -1 & m \end{bmatrix}$$

Corresponding elements at row 1 and column 1:

$$-1 = \frac{-2}{-2m + 6}$$
$$\begin{aligned} 2m - 6 &= -2 \\ m &= 2 \end{aligned}$$

Example 26

It is given that $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and matrix A has an order 2×2. Calculate matrix A.

Solution:

Since the product of $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ with A is an identity matrix, hence A is an inverse matrix of $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$.

$$A = \frac{1}{(1)(8) - (2)(3)}\begin{bmatrix} 8 & -2 \\ -3 & 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 8 & -2 \\ -3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Self Practice 2.2f

1. For each of the following matrices, determine whether the inverse matrix exists. If yes, calculate the inverse matrix.

(a) $\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 5 \\ 3 & -9 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

2. Calculate the inverse matrices for the following matrices.

(a) $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & -5 \\ 2 & 7 \end{bmatrix}$

3. It is given that matrix $G = \begin{bmatrix} 2 & 1 \\ 3 & p \end{bmatrix}$. Calculate the value of p if

(a) matrix G does not have an inverse matrix,

(b) $G^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$.

4. It is given that $\begin{bmatrix} 4 & 10 \\ \frac{1}{2} & 1 \end{bmatrix} P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and matrix P has an order 2×2. Calculate matrix P.

How to use matrix method to solve simultaneous linear equations?

Learning Standard

Use the matrix method to solve simultaneous linear equations.

Simultaneous linear equations can be solved using the matrix method as shown in the steps below.

Simultaneous linear equations

$ax + by = p$

$cx + dy = q$

Matrix form $AX = B$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

where a, b, c, d, p and q are constants while x and y are unknowns

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

Example 27

Write the simultaneous linear equations below in the form of matrix.

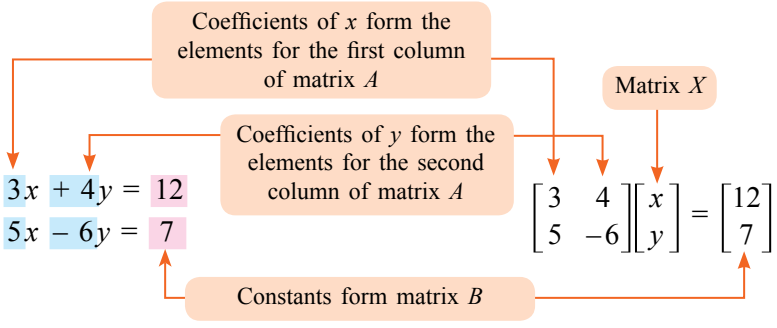
$$3x + 4y = 12$$
$$5x - 6y = 7$$

Critical Mind

Is this multiplication feasible?

$AA^{-1}X = BA^{-1}$

Solution:



Critical Mind

If the equations in Example 27,
$5x - 6y = 7$
$3x + 4y = 12$
are written in the form of a matrix, does the arrangement of the equations affect the answer?

The simultaneous linear equations can be written as $\begin{bmatrix} 3 & 4 \\ 5 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$.

Example 28

Solve the simultaneous linear equations below using the matrix method.

$$x - 2y = 1$$
$$3x - 4y = 4$$

Solution:

$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ ← Write the simultaneous linear equations in the form of matrix

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(1)(-4) - (-2)(3)}\begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$$

Hence, $x = 2$ and $y = \frac{1}{2}$ ← Final answer

Smart TIPS

Solving simultaneous linear equations means calculating the values of x and y. Therefore, the final answer states the values.

Checking Answer

Multiply $\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 - 1 \\ 6 - 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Self Practice 2.2g

1. Write the simultaneous linear equations below in the form of matrix.

(a) $x - y = 7$, $x + 3y = 5$
(b) $3x + y = 0$, $5x + 2y = -14$
(c) $7x + 2y = -11$, $2x - y = -10$
(d) $3x + 2y - 14 = 0$, $4y = 5x - 5$
(e) $2x + y + 4 = 0$, $y - 3x = 11$
(f) $2x + y = -9$, $5x = -12$
(g) $2x = 5y$, $\frac{x}{5} + 2y = 3$
(h) $\frac{x}{y} = 4$, $0.8(x + 5) = 3y$

2. In a chess competition, the total number of participants is 100. The number of male participants, x, is 14 fewer than 2 times the number of female participants, y. Write the simultaneous linear equations that represent the above information in the form of matrix.

3. Solve the simultaneous linear equations below using the matrix method.

(a) $x - 2y = 5,\ 2x - 3y = 10$
(b) $2x - 5y = 1,\ 3x - y = -5$
(c) $2x - y = 8,\ x + y = 1$
(d) $3x + 2y = 4,\ 9x + 4y = 14$
(e) $4x + 3y = 11,\ 2y = 9 - 6x$
(f) $5x - 5y - 6 = 0,\ 2x - 2.1 = 3y$
(g) $p + 3q = 4,\ 3 + \frac{p}{2} = q$
(h) $m + n = 5,\ \frac{m}{2} - \frac{n}{4} = 1$

How to solve problems involving matrices?

Learning Standard

Solve problems involving matrices.

Write linear equations in the form of $ax + by = p$, $cx + dy = q$, where a, b, c, d, p and q are constants while x and y are unknowns → Write the simultaneous linear equations in the form of matrix $AX = B$ → Solve with multiplication with inverse matrix: $X = A^{-1}B$

Example 29

Based on the conversation above, how much is the price of a child ticket and an adult ticket?

Solution:

Understanding the problem

The price of 2 child tickets and 1 adult ticket is RM32.
The price of 5 child tickets and 3 adult tickets is RM88.

x = the price of a child ticket
y = the price of an adult ticket

Devising a strategy

(a) Form two linear equations.
(b) Express the equations in the form of matrix and solve it.

Implementing the strategy

$$2x + y = 32$$
$$5x + 3y = 88$$
$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 32 \\ 88 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(2)(3) - (1)(5)}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 32 \\ 88 \end{bmatrix}$$
$$= \frac{1}{6 - 5}\begin{bmatrix} 96 - 88 \\ -160 + 176 \end{bmatrix}$$
$$= \begin{bmatrix} 8 \\ 16 \end{bmatrix}$$

Making a conclusion

The price of a child ticket is RM8 and an adult ticket is RM16.

Example 30

	Camera K	Camera L
Assembling	10 minutes	10 minutes
Packaging	5 minutes	9 minutes

Syarikat Komunikasi Era Baru produces two camera models, K and L. Each camera produced needs to go through two departments. They are the Assembling Department and the Packaging Department. The table above shows the time for assembling and packaging for each type of camera. It is given that the Assembling Department operates for 12 hours a day and the Packaging Department operates for 9 hours a day. Calculate the number of camera K and camera L produced in a day.

Solution:

Understanding the problem

The total time for assembling is 12 hours, which is equivalent to 720 minutes.

The total time for packaging is 9 hours, which is equivalent to 540 minutes.

x = number of camera K produced

y = number of camera L produced

Devising a strategy

(a) Form two linear equations.

(b) Express the equations in the form of matrix and solve it.

Implementing the strategy

$$10x + 10y = 720$$
$$5x + 9y = 540$$

$$\begin{bmatrix} 10 & 10 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 720 \\ 540 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(10)(9) - (10)(5)} \begin{bmatrix} 9 & -10 \\ -5 & 10 \end{bmatrix}\begin{bmatrix} 720 \\ 540 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 1080 \\ 1800 \end{bmatrix}$$

$$= \begin{bmatrix} 27 \\ 45 \end{bmatrix}$$

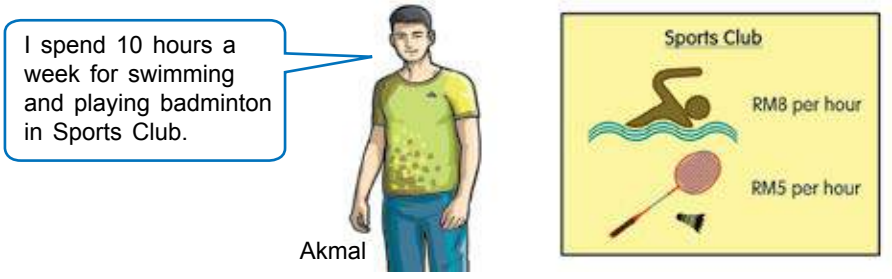
Making a conclusion

The number of camera K produced is 27 units and the number of camera L produced is 45 units.

Self Practice 2.2h

1. A research is conducted on the sales of two types of curry puffs with sardine and potato fillings. In the first hour, 24 curry puffs with sardine filling and 18 curry puffs with potato filling are sold, and the total amount of sales is RM28.80. In the next hour, 30 curry puffs with sardine filling and 14 curry puffs with potato filling are sold, and the total amount of sales is RM29.20. Calculate the prices for a curry puff with sardine filling and a curry puff with potato filling using the matrix method.

2. Akmal spends RM68 a week on the two sports as mentioned below. Calculate the time, in hours, Akmal spends in the Sports Club for swimming and playing badminton in a week using the matrix method.



3. Puan Komala and Puan Lily go to the market to buy papaya and banana. The table below shows the mass of the papaya and banana bought by them.

	Papaya	Banana
Puan Komala	4 kg	2 kg
Puan Lily	5 kg	3 kg

Puan Komala and Puan Lily pay RM26 and RM35 respectively for the two types of fruits. Calculate the prices for one kilogram of papaya and one kilogram of banana using the matrix method.

4. A building has several parking spaces for cars and motorcycles. One day, there were a total of 66 vehicles parked there and the total number of wheels was 190. Calculate the number of cars and the number of motorcycles parked that day using the matrix method. Assume that all motorcycles are two-wheeled.

5. Encik Jefri and Encik Tan invest in Unit Trust P and Unit Trust Q as shown in the table below.

	Unit Trust P	Unit Trust Q
Encik Jefri	RM5 000	RM3 000
Encik Tan	RM6 000	RM4 000

After a year, Encik Jefri gets a dividend of RM350 from the investment of both the unit trusts while Encik Tan gets a dividend of RM440. Calculate the dividend rates of Unit Trust P and Unit Trust Q using the matrix method.

Summary Arena

MATRICES

Matrices
- A set of numbers arranged in rows and columns to form a rectangular or a square array
- Written inside bracket [ ] or ( )

Order
Order $m \times n$ has m rows and n columns

Element
a_{ij} is the element of i^{th} row and j^{th} column

Equal matrices
$A = B$ if the order of both the matrices are the same and the corresponding elements are equal

Basic Operation on Matrices

Add and subtract matrices
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

Multiply a matrix by a number
$$n\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} na & nb \\ nc & nd \end{bmatrix}$$

Multiply two matrices
$$\begin{matrix} A & B & = & AB \\ m \times n & n \times p & & m \times p \end{matrix}$$

$$[a \quad b]\begin{bmatrix} c \\ d \end{bmatrix} = [ac + bd]$$

$$\begin{bmatrix} c \\ d \end{bmatrix}[a \quad b] = \begin{bmatrix} ca & cb \\ da & db \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Identity matrix, I,
order $n \times n$ with element 1 along the main diagonal and the other elements are 0

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$AI = IA = A$$

Inverse matrix
Inverse matrix of A is denoted by A^{-1}.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$AA^{-1} = A^{-1}A = I$$

Solving simultaneous linear equations

Simultaneous linear equations
$ax + by = p$
$cx + dy = q$

Matrix form $AX = B$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$
where a, b, c, d, p and q are constants while x and y are unknowns

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

Reflection

At the end of this chapter, I can

represent information from real situations in the form of matrices.		
determine the order of a matrix, hence identify certain elements in a matrix.		
determine whether two matrices are equal.		
add and subtract matrices.		
multiply a matrix by a number.		
multiply two matrices.		
explain the characteristics of identity matrix.		
explain the meaning of inverse matrix and hence determine the inverse matrix for a 2×2 matrix.		
use the matrix method to solve simultaneous linear equations.		
solve problems involving matrices.		

MINI PROJECT

Transportation companies use network to represent the routes of their transports. Network consists of vertices that are connected with edges.

In the diagram below, the vertices P, Q, R, S and T represent the cities while the edges represent the routes of bus between two cities. All the routes are roads that connect adjacent cities. This route system can be represented with the matrix as below.

$$\text{From} \begin{matrix} \\ P \\ Q \\ R \\ S \\ T \end{matrix} \begin{matrix} \text{To} \\ \begin{matrix} P & Q & R & S & T \end{matrix} \end{matrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Prepare a report about the route system of buses (or any other types of transportation) in your area. Your report should consist of

(i) the introduction of the public transport system in your area,
(ii) the use of matrix to represent the route system of the public transport,
(iii) the meaning of the elements in the matrix.

Extensive Practice

Scan the QR code or visit bit.do/QuizE02 for interactive quiz

UNDERSTAND

1. State the number of rows and the number of columns of the matrix $\begin{bmatrix} 9 & -2 \\ 1 & 6 \\ 5 & 7 \end{bmatrix}$.

2. It is given that $A = \begin{bmatrix} 3 & 2 \\ 0 & -1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $AB = C$. Determine the order of matrix C.

3. It is given that matrix $D = \begin{bmatrix} 4 & p \\ -2 & 3 \end{bmatrix}$. Calculate the value of p if the determinant of matrix D is 0.

4. Given that matrix $E = \begin{bmatrix} -1 & 2 \\ 5 & 4 \end{bmatrix}$, show that $E + E + E = 3E$.

5. Write the following simultaneous linear equations in the form of matrix.

$$\begin{aligned} m - 3 &= 4n \\ 3m + 2n - 2 &= 0 \end{aligned}$$

MASTERY

6. It is given that $G = \begin{bmatrix} 4 & r \\ 1 & 2 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 3 \\ -1 & s \end{bmatrix}$. Calculate the values of r and s if $GH = HG$.

7. It is given that $A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$ and $AB = I$. Calculate matrix B.

8. Given that $P = \begin{bmatrix} 10 & -5 \\ -2 & 1 \\ 0 & 2z - 3 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & y \\ 0.2 & -\frac{1}{3} \\ 8 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 11 & -25 \\ y + 6x & -0.2 \\ 24 & 9 \end{bmatrix}$ and $0.8P + 3Q = R$, calculate the values of x, y and z.

9. It is given that matrix $F = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$.

(a) Calculate F^3.

(b) Hence, calculate matrix G if $F^3 - 5G = 12F$.

10. Given that $\frac{1}{p}\begin{bmatrix} 2 & q \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, calculate the values of p and q.

11. (a) Write the simultaneous linear equations below in the form of matrix.

(b) Calculate the values of x and y using the matrix method.

$$\begin{aligned} 2y - x &= 5 \\ 3x - 8y &= -19 \end{aligned}$$

CHALLENGE

12. A marathon race has 128 participants. The number of male participants is 16 fewer than 2 times the number of female participants. Calculate the number of male participants and female participants in the marathon race using the matrix method.

13. It is given that the simultaneous linear equations $px + 4y = 10$ and $qx - 2y = 1$ have no solution. Express p in terms of q.

14. Faris took a course in a college. He registered for three subjects in the first semester. The overall mark for each subject was calculated based on the marks of coursework and exam according to the percentage of each section. Table 1 shows the marks obtained by Faris in the first semester. Table 2 shows the weightage given to coursework and exam.

	Coursework	Exam
Mathematics	80	70
English	60	75
Computer Science	74	84

Table 1

	First Semester
Coursework	60%
Exam	40%

Table 2

(a) Represent the information in Table 1 and Table 2 with matrices.
(b) Calculate the overall marks for Mathematics in the first semester using the matrix method.
(c) Determine the best performed subject in the first semester.

15. Syahirah is undergoing a diet plan involving two types of drink, P and Q. The table below shows the amount of protein and calories for a glass of the drinks.

	Drink P	Drink Q
Protein (g)	6	4
Calories (kcal)	95	110

The diet plan suggests that Syahirah consume a total of 16 g of protein and get 300 kcal per day from the two types of drinks.
(a) Form two linear equations with the above information.
(b) Calculate the number of glasses of drink P and drink Q Syahirah needs to consume daily according to the diet plan using the matrix method.

16. Mr Sanjay sold two brands of air conditioners, K and L. The prices of the air conditioners K and L are RM1 500 and RM2 000 respectively. The commissions for selling an air conditioner K and an air conditioner L are 3% and 4% respectively. In the month of May, Mr Sanjay sold 50 units air conditioners and received a total commission of RM2 880. Calculate the number of air conditioners K and L sold using the matrix method.

EXPLORING MATHEMATICS

Cryptography is a science of information safety. It involves techniques such as combining words into the form of images or writing words in secret codes so that the words cannot be read by a third party. During World War II, the German army used the Enigma machine to write their secret messages. Three Poland mathematicians managed to decrypt the messages from the Enigma machine and assisted the Allied Powers to end the war.

Use the code system below, send a message "GURU KELAS" to your friend.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
O	P	Q	R	S	T	U	V	W	X	Y	Z	!	?	.
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29

Follow the following steps.

(a) Write the message in a few matrices with order 2×1.

(b) Use the lock $M = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ to encrypt the message, that is, matrix M is multiplied with every matrix formed in (a).

(c) The products obtained will be transformed into a secret message in alphabets based on the code system above. If the product is a negative number, add the product with 30. Send the secret message to your friend.

(d) When receiving the secret message, your friend needs to decrypt the message based on the following steps:

(i) write the secret message obtained in a few matrices with order 2×1.

(ii) multiply the key $K = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ with every matrix formed in (d)(i).

(iii) the products obtained are transformed into the original message by referring to the code system above. If the product is a negative number, add the product with 30.

Tips for the steps:

(a) For example, message "DI BAS", alphabets "D" = 4, "I" = 9, " " = 0, "B" = 2, "A" = 1 and "S" = 19. Hence the matrices formed are $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 19 \end{bmatrix}$.

(b) For example,
$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \end{bmatrix}$,
$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and
$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 19 \end{bmatrix}$.

(c) For example,
$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.
Therefore, $\begin{bmatrix} -1 + 30 \\ 4 \end{bmatrix} = \begin{bmatrix} 29 \\ 4 \end{bmatrix}$.
By referring to the code system, $\begin{bmatrix} 29 \\ 4 \end{bmatrix}$ will be represented with $\begin{bmatrix} . \\ D \end{bmatrix}$.
A full secret message produced is ".D? MA".

CHAPTER 3

Consumer Mathematics: Insurance

What will you learn?

- Risk and Insurance Coverage

Why study this chapter?

The knowledge on risk and insurance as a financial protection is very important for every individual as a preparation for the future. It is also very important in personal financial management because we might use a part of our income for insurance.

Do you know?

In Malaysia, the insurance industry is controlled and regulated by Bank Negara Malaysia under the Insurance Act 1996 which replaces the Insurance Act 1963. This act complies with the regulations of insurance and serves as basic legislation that controls the insurance business in Malaysia. Apart from this, this act empowers Bank Negara Malaysia to determine matters in accordance with the provisions of the act.


UNDANG-UNDANG MALAYSIA

Akta 553

Akta Insurans 1996

For more information:

bit.do/DoYouKnowChap3

WORD BANK

deductible	*deduktibel*
general insurance	*insurans am*
life insurance	*insurans hayat*
rate	*kadar*
co-insurance	*ko-insurans*
coverage	*perlindungan*
policy	*polisi*
premium	*premium*
risk	*risiko*

We often come across news and headlines in the media of families being affected by disasters such as flood, fire, theft and accidents. Have you ever thought that you and your family might face such a disaster in the future? If so, is your family ready for it?

If it happens to us, we would definitely have to bear the financial losses. Therefore, it is very important for you to understand the importance of insurance because insurance helps us to face uncertainties and losses.

3.1 Risk and Insurance Coverage

What is the meaning of risk and the importance of insurance coverage?

Learning Standard

Explain the meaning of risk and the importance of insurance coverage, hence determine the types of life insurance and general insurance for protecting a variety of risks.

Risk

Do you know that every one of us is exposed to danger that might happen to us, our family or our assets?

Info Bulletin

Important elements in the definition of risk:
(i) Risk is unpredictable
(ii) Risk involves loss

A road accident that injures the driver and passengers, damage of property in a fire, expensive medical bills due to critical illness and others may occur in life. It is certain that these incidents cause us to bear some losses. The fact that we are exposed to such losses and the uncertainty of whether these incidents could happen to us, create a risk in our lives. Therefore, in general, **risk** is the possibility of a disaster that cannot be avoided.

As a pupil, facing risks is unavoidable. The risks that you might face are meeting an accident on the way to school, getting injured in a football game or falling down the stairs while going to the library.

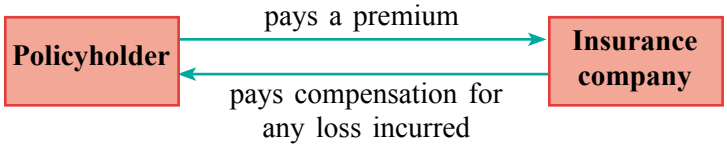
In spite of being careful and practising a healthy lifestyle, there are still some risks you might face. Hence, with insurance that is available to all ages, you will be given a financial protection if unfortunate events happen to you.

Critical Mind

Which of the following is the risk that can be measured by money? Discuss.
(i) The risk in buying a second-hand car.
(ii) The risk in investing in stock market.
(iii) The risk of a motorcycle being stolen.

Insurance

Insurance is intended to transfer risks from individuals to insurance organisations. It is a contract signed between the insurance company and the insurance owner. Under this contract, the insurance company promises to pay compensation for the loss covered in the policy, in return for the premium paid by the policyholder.



The amount of compensation payable to the policyholder is only the amount of the loss suffered. This is because the key principle in the insurance system is to restore the financial position of the policyholder to the pre-loss condition, which is also known as the **principle of indemnity**. Therefore, the insurance principle will not allow the policyholder to gain profit from the insurance purchased.

Insurance cannot prevent a loss from occurring but can help to reduce financial burden borne by a policyholder in the occurrence of losses or accidents. This is because buying an adequate sum of insurance removes the uncertainty of financial loss in the occurrence of the loss insured.

Info Bulletin

Insurance company
The party that agrees to pay compensation for the losses of the agreed terms.

Policyholder (Insurance owner)
The individual who will claim and receive compensation for the losses.

Insurance contract
A policy and an evidence of an agreement made between the insurance company and the policyholder.

Premium
An amount of money payable by the policyholder to the insurance company.

Info Bulletin

Principle of indemnity:
Insurance companies will pay compensation to policyholders in the occurrence of a loss insured for an amount not exceeding the loss incurred, subject to the sum insured.

What are life insurance and general insurance?

Insurance can be divided into two main types, namely life insurance and general insurance. Both of these insurances protect different risks.

MIND MOBILISATION 1 Group

Aim: To determine the importance of insurance to cover various types of risks.

Steps:

1. Divide the class into several groups.
2. Each group has to search newspaper articles or online news related to disasters that have befallen an individual or a family.

3. Based on the situation in the news, identify the types of losses suffered and types of risks covered by insurance.

Situation	Type of loss suffered	Type of risk covered by insurance

4. Present your findings in a Gallery Walk.

Discussion:
Why is insurance important to policyholder?

The results of Mind Mobilisation 1 show that insurance is important in protecting different types of risks from accidental losses and can cause the policyholder to bear a financial loss.

Life insurance

Life insurance guarantees the payment of a stated amount of monetary benefits to the policyholder upon the death of the insured, or under other circumstances specified in the contract. The risks covered by life insurance are as follows:

The purpose of life insurance is to provide financial protection to family members who depend on the policyholder, when the policyholder dies. However, if the policyholder is alive but suffers from a total and permanent disability, the policyholder may receive compensation from the insurance company in accordance with the terms set out in his/her life insurance policy.

Application & Career

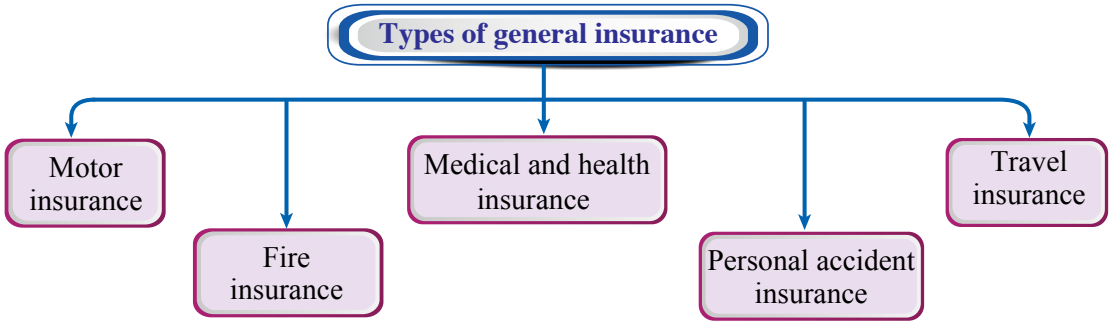
Actuary is an expert in managing and calculating risks in premium of insurance policy, annuity rate and others for an insurance company.

Interactive Platform

Will compensation be made if there is no loss incurred? Discuss.

General insurance

General insurance provides coverage against any loss or damage incurred, apart from the risks covered by life insurance. There are many types of general insurance that cover a variety of risks.



(i) Motor insurance

Motor insurance provides coverage against any loss or damage related to the use of a motor vehicle. The table below shows the four types of motor insurance policies that cover your vehicle.

Info Bulletin

Motor insurance is a compulsory insurance that every motor vehicle in Malaysia must have under the Road Transport Act 1987.

Coverage \ Policy	Act	Third party	Third party, fire & theft	Comprehensive
Liability to third party due to injury and death.	Yes	Yes	Yes	Yes
Loss of property suffered by third party.	No	Yes	Yes	Yes
Loss to own vehicle due to accidental fire or theft.	No	No	Yes	Yes
Loss and damage to own vehicle due to accident.	No	No	No	Yes

A comprehensive policy provides a thorough protection compared to the other three policies as above. The similarity on these policies is to give a protection excluding the claims by the driver and passengers such as injury and death.

As an example, Agus was driving his car, with his friend, Faizal as a passenger. Agus lost control and knocked into another car driven by Devi. Agus, Faizal and Devi were injured, and Agus and Devi's cars were damaged.

Info Bulletin

In motor insurance,
- the first party → the driver (policyholder),
- the second party → the insurance company
- the third party → any other individuals that are involved in a accident caused by the driver, excluding the passenger

If Agus insured his car under the comprehensive policy, this policy will cover injury and car damage claimed by the third party, who is Devi and the loss and damage of Agus's car. However, the medical costs incurred by Agus and Faizal will not be covered.

Info Bulletin

Liability is the amount of debt that needs to be paid with money, goods or service.

(ii) Fire insurance

Fire insurance provides coverage against damage caused by fire, lightning and explosion that occur at home or business premise. Compensation will be paid by the insurance company to recover the policyholder's financial position, subject to the sum insured. For coverage other than fire, lightning and explosion, policyholder may incorporate additional coverages such as hurricane, flood, riot and others into the existing fire policy at an additional premium.

(iii) Medical and health insurance

Apart from the risk of death, you should also be aware of the risks of deteriorating health. This is because you may have to bear medical expenses such as hospitalisation and surgery costs. Some of the policies in this insurance are:

- hospitalisation and surgical insurance
- critical illness insurance
- disability income insurance
- hospital income insurance

Info Bulletin

Scan the QR code or visit bit.do/Chap3MHI for more information about medical and health insurance.

(iv) Personal accident insurance

Personal accident insurance provides coverage in the occurrence that the policyholder suffers a bodily injury, disability or death resulting directly from accident. This insurance is different from life insurance, and medical and health insurance.

Interactive Platform

Why is the medical and health insurance important even though the government provides medical and health services in government hospitals? Discuss.

(v) Travel insurance

Travel insurance protects policyholder against losses during travel whether by land, air or sea such as death and permanent disability, loss of luggage, passport and money, medical expenses and others.

The coverage period for most general insurance policies is one year or less than a year. The premiums are paid once only when you need the coverage of the policy.

Group insurance

Group insurance provides coverage to a group of individuals, typically employees in a company or pupils in schools and students in educational institutions.

- **Group insurance for organisation**
  Financial protection is provided to employees of an organisation in the events of death, disability, hospitalisation and surgery under certain policy and coverage limit. With this insurance, the employees of the company will enjoy proper protection from their employer.

- **Group insurance for pupils**
  Financial protection is provided to pupils in the events of death, paralysis, disability and wheelchair allowance under certain policy and coverage limit. For example, Ministry of Education Malaysia has implemented a protection scheme for pupils of government schools and government-aided schools under the *Takaful Pelajar Sekolah Malaysia* (TPSM) scheme.

Info Bulletin

mySalam
Perlindungan Masyarakat

The mySalam national health protection scheme is an initiative started by the government which aims to provide 8 million Malaysian citizens with free takaful protection. Visit mysalam.com.my for more information.

Generally, the reasons you should get an insurance are as follows.

- ✓ As a financial aid to the family in the occurrence of your disability, critical illness or death
- ✓ Managing living expenses, debts and commitments in the event that you are unable to work
- ✓ Paying for high medical expenses
- ✓ As compensation for losses incurred

Self Practice 3.1a

1. Based on the situation below, answer the following questions.

> Encik Daud buys an insurance policy to cover himself in case of an accident from *Syarikat Insurans Bersatu* for RM300 000 with a monthly payment of RM100.

(a) Who is the insurance company and the policyholder?
(b) How much is the coverage limit?
(c) How much is the monthly premium?
(d) State the risk to be insured.

2. One day, Melisa's car collided with another car in an accident and she was at fault. Both cars were badly damaged, and the other driver suffered a broken leg. Melisa wants to file a claim for the damage for both cars, as well as the medical expenses for the driver of the other car.

Based on the scenario above, state the claim that Melisa can make from her insurance company if she insures her car under the third party policy.

How to investigate, interpret and perform calculations involving insurance rates and premiums?

Learning Standard

Investigate, interpret and perform calculations involving insurance rates and premiums.

Example 1

The table below shows the premiums for travel insurance offered by *Syarikat Insurans* PQ *Bhd.* to countries in Asia and Europe.

Number of days	Policyholder (RM)		Policyholder and spouse (RM)		Family (RM)	
	Asia	Europe	Asia	Europe	Asia	Europe
1 – 5	39	53	69	98	87	133
6 – 10	58	79	107	150	136	184
11 – 18	79	127	152	246	218	304
Annual premium (18 – 69 years old)	230	280	–	–	–	–

(a) What are the factors that influence the difference in premiums for the travel insurance?

(b) Why is the premium higher for a longer travel period?

(c) Shahir's work as a photographer requires him to visit many countries in Europe within a year. He will visit a country for 12 to 15 days. Which insurance is suitable for Shahir to buy in relation to his job? Give your reason.

Solution:

(a) Destination, duration of travel and number of people insured.

(b) A longer travel period increases the probability of loss to the policyholder while abroad.

(c) Shahir should buy annual premium travel insurance for countries in Europe as it is more economical than buying insurance for 11 to 18 days per trip.

How to calculate a premium for life insurance?

For life insurance, the amount of premium paid depends on the chosen face value, age, gender and smoking status. This premium calculation is based on a premium rate schedule for every RMx face value.

$$\text{Premium} = \frac{\text{Face value of policy}}{\text{RM}x} \times \left(\begin{array}{c}\text{Premium rate}\\ \text{per RM}x\end{array}\right)$$

Info Bulletin

(i) The premium rate schedule provides the gross amount of premium charged to the policyholder.

(ii) The face value of policy is the amount of money chosen by the policyholder to obtain insurance coverage. This amount will be paid to the beneficiaries in the occurrence of death or total and permanent disability of the policyholder on the policy's maturity date.

Example 2

The annual premium rate schedule per RM1 000 face value of a yearly renewable term insurance offered by *Syarikat Insurans* XYZ is as follows.

Age	Male (RM)		Female (RM)	
	Non-smoker	Smoker	Non-smoker	Smoker
35	2.12	2.72	1.45	1.78
36	2.18	2.80	1.50	1.84
37	2.26	2.91	1.56	1.93
38	2.36	3.05	1.63	2.03
39	2.49	3.23	1.71	2.14
40	2.66	3.47	1.80	2.26

Info Bulletin

Yearly renewable term insurance provides protection for one year and can be renewed for the next year as stipulated by the insurance company.

(a) Why does the premium rate increase
  (i) as the age increases?
  (ii) for a smoker?

(b) Based on the table, calculate the annual premium for each of the following situations.
  (i) Mr Guan wants to buy an insurance policy worth RM100 000. He is 39 years old, healthy and a non-smoker.
  (ii) Madam Shapuva is 36 years old, healthy and a non-smoker. She wants to buy an insurance policy worth RM250 000 and wants to add on a critical illness policy. *Syarikat Insurans* XYZ offers a critical illness policy with a coverage of 30% of basic face value and the premium rate is RM1.77 per RM1 000 based on Madam Shapuva's age and health status.

Solution:

(a) (i) Premium rate increases with increasing age because the life expectancy of each person decreases with age.
  (ii) The probability of a smoker being exposed to illness is higher than that of a non-smoker who lives a healthy lifestyle.

(b) (i) From the table, the premium rate is RM2.49.
Mr Guan's annual premium

$$= \frac{\text{RM100 000}}{\text{RM1 000}} \times \text{RM2.49}$$

$$= \text{RM249.00}$$

(ii) From the table, the premium rate is RM1.50.
The total coverage for critical illness

$$= \frac{30}{100} \times \text{RM250 000}$$

$$= \text{RM75 000}$$

Madam Shapuva's annual premium
= Annual basic premium + Annual additional premium for critical illness

$$= \frac{\text{RM250 000}}{\text{RM1 000}} \times \text{RM1.50} + \frac{\text{RM75 000}}{\text{RM1 000}} \times \text{RM1.77}$$

$$= \text{RM375} + \text{RM132.75}$$

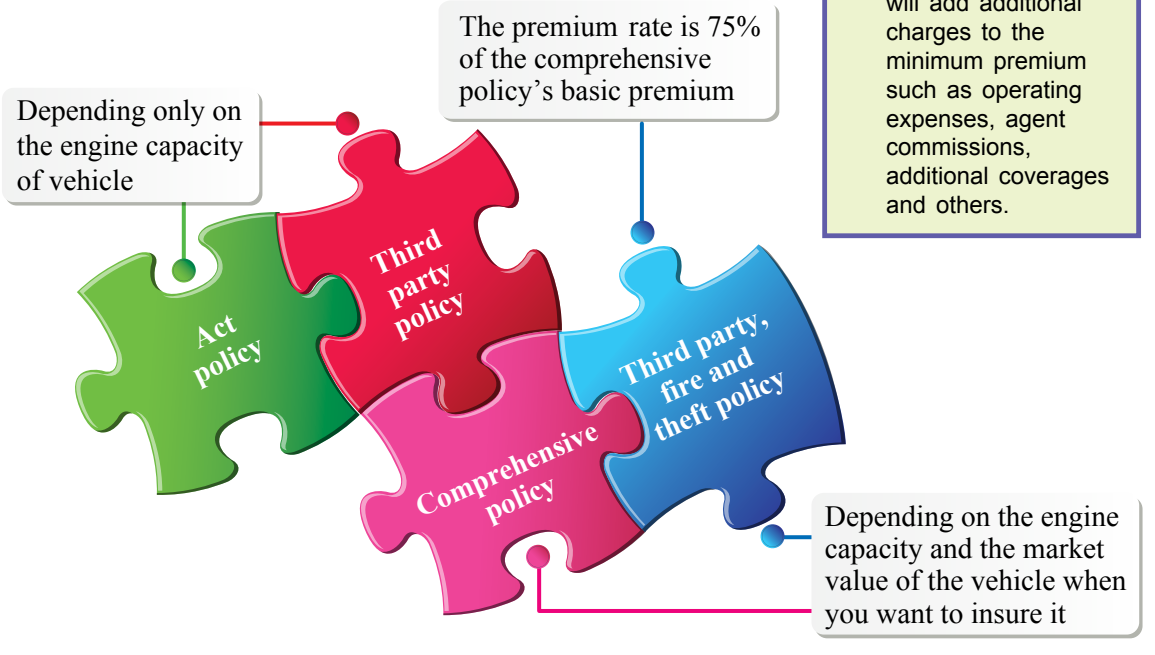
$$= \text{RM507.75}$$

How to calculate a premium for motor insurance?

Motor insurance is calculated based on the Motor Tariff that provides the minimum premium for the insurance coverage. These tariffs are regulated by law. The amount of the premium depends on the factors such as the type of vehicle and its usage, the engine capacity, the type and amount of coverage desired. The premiums charged are different for each motor policy.

Info Bulletin

(i) Tariff is a list of fixed prices used to organise and control premium charges and wording of policies under the insurance act.
(ii) Insurance companies will add additional charges to the minimum premium such as operating expenses, agent commissions, additional coverages and others.



The table below shows the premium rates under the Motor Tariff for motor policies issued in Peninsular Malaysia, Sabah and Sarawak.

Engine capacity not exceeding (cc)	Peninsular Malaysia		Sabah and Sarawak	
	Comprehensive policy (RM)	Third party policy (RM)	Comprehensive policy (RM)	Third party policy (RM)
1 400	273.80	120.60	196.20	67.50
1 650	305.50	135.00	220.00	75.60
2 200	339.10	151.20	243.90	85.20
3 050	372.60	167.40	266.50	93.60
4 100	404.30	181.80	290.40	101.70
4 250	436.00	196.20	313.00	110.10
4 400	469.60	212.40	336.90	118.20
Over 4 400	501.30	226.80	359.50	126.60

* For comprehensive policy, the rate charged is for the first RM1 000 of the sum insured

Source: Schedule of Motor Tariff 2015

The formulae to calculate the basic premium of the comprehensive policy:

(i) For Peninsular Malaysia, the basic premium
= Rate for the first RM1 000 + RM26 for each RM1 000 or part thereof on value exceeding the first RM1 000

(ii) For Sabah and Sarawak, the basic premium
= Rate for the first RM1 000 + RM20.30 for each RM1 000 or part thereof on value exceeding the first RM1 000

i-Technology

Visit the General Insurance Association of Malaysia (PIAM) website for more information on general insurance services in Malaysia.
http://www.piam.org.my/

Example 3

Encik Ramli owns and uses a Proton Exora 1.6 in Peninsular Malaysia. The information of the car is as follows:

Sum insured : RM60 000
Age of vehicle : 5 years
Engine capacity : 1 600 cc
NCD : 25%

Calculate the gross premium for Encik Ramli's car under the comprehensive policy, the third party, fire and theft policy, and the third party policy.

Solution:

For comprehensive policy:

(a) The first RM1 000	RM305.50
(b) RM26 × 59 (each RM1 000 balance)	RM1 534
(c) Basic premium = (a) + (b)	RM1 839.50
(d) NCD 25%	RM459.88
(e) Gross premium = (c) – (d)	RM1 379.62

Refer to the table.

$\frac{60\,000 - 1\,000}{1\,000} = 59$

0.25 × 1 839.50 = 459.88

For third party, fire and theft policy:

(a) Basic premium	RM1 379.63
(b) NCD 25%	RM344.91
(c) Gross premium = (a) – (b)	RM1 034.72

0.75 × 1 839.50 = 1 379.63

0.25 × 1 379.63 = 344.91

For third party policy:

(a) Basic premium	RM135.00
(b) NCD 25%	RM33.75
(c) Gross premium = (a) – (b)	RM101.25

Refer to the table.

0.25 × 135.00 = 33.75

Info Bulletin

No Claim Discount (NCD) clause will be issued if no claim is made against your policy within the period of coverage before the policy renewal is made. As such, the premiums payable can be reduced by your NCD eligibility. You will lose your entire NCD eligibility when a personal or third-party claim is made against your policy. NCD eligibility depends on your vehicle class and the number of years of continuous driving experience without claim.

However, the premiums payable may differ from the actual value of the premiums that the insurance company may charge. There are additional charges which are payable by the policyholder such as commission to the agent, service tax and stamp duty.

Self Practice 3.1b

1. The table below shows some of the benefits of personal accident insurance offered by *Syarikat Insurans* RST.

Benefit	Total coverage (RM)	Annual premium based on occupation	
		Class 1 & 2 (RM)	Class 3 (RM)
Accidental death	100 000	364.75	1 065.40
Total and permanent disability	200 000		
Permanent disability	150 000		

* Class 1 – Occupation in low risk industries and in office.
* Class 2 – Occupation involving non-manual works with semi-hazardous industry.
* Class 3 – Occupation involving manual works and use of tools or machinery.

In your opinion, what makes the premium for class 3 occupation higher than class 1 and class 2?

2. The table below shows the annual premium rate per RM1 000 face value of a renewable life insurance offered by an insurance company.

Age	Male (RM)		Female (RM)	
	Non-smoker	Smoker	Non-smoker	Smoker
27	2.12	2.72	1.18	1.40
28	2.12	2.73	1.19	1.42
29	2.12	2.75	1.21	1.44
30	2.12	2.79	1.23	1.46

Based on the face value of RM120 000,

(a) calculate the annual premium for a 29-year-old man who smokes and a 29-year-old man who does not smoke.

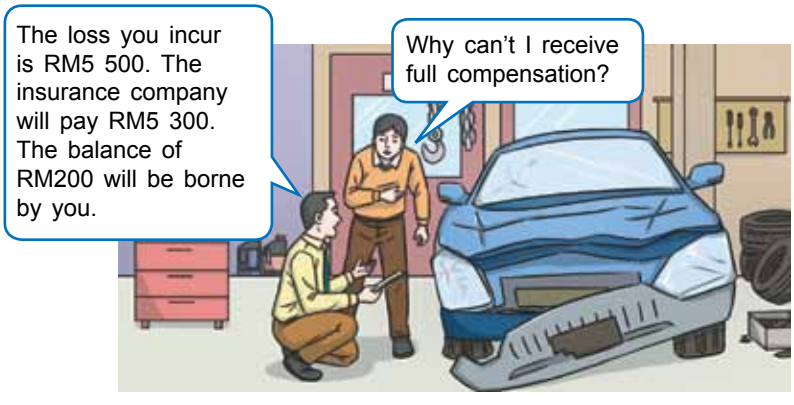
(b) calculate the annual premium for a 30-year-old woman who smokes and a 30-year-old woman who does not smoke.

3. Encik Zakri stays in Peninsular Malaysia. He wants to buy a motor insurance policy. The following is the information regarding the vehicle he wants to insure.

Sum insured	: RM85 000
Age of vehicle	: 6 years
Engine capacity	: 2 494 cc
NCD	: 30%

Calculate the gross premium for the comprehensive policy, the third party, fire and theft policy, and the third party policy based on the Schedule of Motor Tariff 2015.

How to solve problems involving insurance?



There are several provisions in the insurance contract that may cause the policyholder to receive compensation less than the actual loss and hence have to bear part of the loss. Amongst these provisions are **deductible** and **co-insurance**, which is also known as joint insurance.

Learning Standard

Solve problems involving insurance including deductible and co-insurance.

Smart TiPS

Read the insurance policy contract carefully before you agree to buy the insurance policy and get a clearer explanation from your insurance agent.

What is deductible?

Deductible is a an amount that must be borne by the policyholder before they can make a claim from the insurance company. Deductible is commonly found in the contract of property insurance, medical and health insurance and motor insurance. This provision is not in the life insurance and personal liability insurance.

Example 4

Madam Suhaila has purchased motor insurance for her car with a deductible provision of RM300. During the insurance coverage period, Madam Suhaila suffered three accidents. The losses suffered in March, July and August were RM2 800, RM250 and RM400 respectively. State whether Madam Suhaila can make a claim for the losses suffered. If yes, state the amount of compensation that can be claimed for each loss.

Solution:

Month	Loss (RM)	Can make a claim?	Amount of compensation (RM)
March	2 800	Yes	2 500
July	250	No	None
August	400	Yes	100

The amount of the loss exceeds the deductible amount. Hence, a claim can be made.
Amount of compensation
= RM2 800 – RM300
= RM2 500

The amount of the loss is less than the deductible amount. Hence, a claim cannot be made for this loss.

The amount of the loss exceeds the deductible amount. Hence, a claim can be made. Amount of compensation = RM400 – RM300 = RM100

In motor insurance, the policyholder is responsible for a compulsory deductible of RM400 if the insured vehicle is driven by an individual who is not named in the policy, who is named in the policy but is under 21 years old, and who is the holder of a provisional (L) driving licence or the holder of a full driving licence of less than 2 years. Other deductibles are at the discretion of the insurance company.

Example 5

Cher Lin has a medical insurance policy with a deductible of RM30 000 per year with an annual limit of RM300 000. In the first year of her insured period, Cher Lin had been hospitalised for appendix surgery and the treatment cost was RM8 000. In the following year, Cher Lin underwent heart surgery and the treatment cost was RM210 000. State the amount borne by Cher Lin and the amount paid by the insurance company for the first and the second year of her insurance period.

Solution:

First year:

Treatment cost = RM8 000

The treatment cost is less than the deductible amount.

Amount borne by Cher Lin = RM8 000

Second year:
Treatment cost = RM210 000 ← The treatment cost is more than the deductible amount.

Amount covered by Cher Lin = RM30 000
Amount paid by the insurance company
= RM210 000 – RM30 000 ← Treatment cost – deductible
= RM180 000

What is co-insurance in property insurance?

Co-insurance is the cost sharing of the loss between the insurance company and the policyholder.

For the co-insurance clause in property insurance, the policyholder is required to insure his property at a certain amount based on the percentage of co-insurance determined by the insurance company from the insurable value of the property. If this co-insurance provision is not met, the policyholder will have to bear some of the losses together with the insurance company.

Therefore, if the policyholder wishes to recover full compensation for the partial loss incurred, he must insure the property in accordance with his co-insurance provisions.

Info Bulletin

(i) Clause is a part of a written legal document.
(ii) The insurable value of property is the actual value of assets, the cost of replacement of the property or any other value stated in the valuation clause of insurance policy.

Amount of required insurance
= Percentage of co-insurance × Insurable value of property

1 **If the insured value = amount of required insurance**

Amount of compensation
= Amount of loss – Deductible
where the amount of loss ≤ the amount of insurance purchased.

2 **If the insured value < amount of required insurance**

Amount of compensation

$$= \left(\frac{\text{Amount of insurance purchased}}{\text{Amount of required insurance}} \right) \times \left(\text{Amount of loss} \right) - (\text{Deductible})$$

3 **Suffered a total loss**

Amount of compensation

$$= \left(\text{Amount of insurance purchased} \right) - (\text{Deductible})$$

Smart TiPS

With deductible provision and co-insurance in insurance policy, the premium becomes lower.

Example 6

Encik Ismail wants to buy fire insurance for his house. The insurable value of the house is RM350 000. The fire insurance policy he wants to buy has a co-insurance provision to insure 80% of its insurable value and a deductible of RM2 000.

(a) Calculate the amount of insurance required by Encik Ismail for the house.

(b) Encik Ismail's house caught on fire and the amount of loss is RM25 000. Calculate the amount of compensation that Encik Ismail will receive if he insures his house
   (i) at the amount of required insurance,
   (ii) at a sum of RM150 000. Hence, calculate the co-insurance penalty.

(c) Encik Ismail's house suffered a total loss. If he insured his house at a sum of RM200 000, calculate the amount compensation he will receive.

Info Bulletin

The amount that policyholders bear on the partial loss incurred as a result of not meeting the co-insurance provision is known as the co-insurance penalty.

Solution:

(a) Amount of required insurance $= \frac{80}{100} \times \text{RM350 000}$
$= \text{RM280 000}$

(b) (i) Amount of compensation $= \text{RM25 000} - \text{RM2 000}$
$= \text{RM23 000}$

(ii) RM150 000 $<$ RM280 000. The sum insured is less than the amount of required insurance.

$$\text{Amount of compensation} = \frac{\text{RM150 000}}{\text{RM280 000}} \times \text{RM25 000} - \text{RM2 000}$$
$$= \text{RM13 392.86} - \text{RM2 000}$$
$$= \text{RM11 392.86}$$

$$\text{Co-insurance penalty} = \text{RM25 000} - \text{RM13 392.86}$$
$$= \text{RM11 607.14}$$

Co-insurance penalty is equivalent to 46.4% of the total loss

(c) Amount of compensation $= \text{RM200 000} - \text{RM2 000}$
$= \text{RM198 000}$

Amount of loss borne by Encik Ismail
= RM350 000 – RM198 000
= RM152 000

If the policyholder insures his property with a higher amount than the amount of required insurance under the co-insurance provision, the compensation calculated using the co-insurance formula will exceed the amount of actual loss incurred. However, the insurance company will only pay the compensation not exceeding the amount of actual loss and the maximum payment for a loss will be equal to the total face value of the insurance purchased by the policyholder.

What is co-insurance in health insurance?

In health insurance contracts, co-insurance is stipulated by the percentage participation clause, particularly for the major medical insurance policy. In this clause, policyholder is required to bear a portion of the medical costs covered by the contract at an agreed rate after taking into account the deductible provisions, if any. For example, 80/20 co-insurance percentage participation means that the insurance company will bear 80% of the medical costs covered by the policy and 20% will be borne by the policyholder.

Example 7

Madam Chen has a major medical insurance policy with a deductible provision of RM500 and a 75/25 co-insurance percentage participation clause in her policy. Calculate the cost borne by the insurance company and Madam Chen herself if the medical cost covered by her policy is RM20 600.

Solution:

Medical cost after deductible
= RM20 600 – RM500
= RM20 100

The cost borne by the insurance company
$= \frac{75}{100} \times$ RM20 100
= RM15 075

The cost borne by Madam Chen
$= \frac{25}{100} \times$ RM20 100 + RM500 ← Deductible
= RM5 525

Guidelines to choose your best insurance:

- Find out the total coverage needed.
- Understand the scope of the coverage, the terms and conditions of the policy.
- Compare the premium rates and insurance benefits.
- Avoid unnecessary coverage.

Smart TiPS

The deductible provision must be met by the policyholder before she can receive the benefits of the insurance she has purchased.

Smart TiPS

The difference between deductible and co-insurance:
- Deductible is an amount the policyholders must pay in advance regardless of the amount of eligible benefits.
- Co-insurance is a cost sharing where the policyholders borne a certain percentage of the loss with the insurance company.

Critical Mind

Why is there no penalty for health insurance?

Example 8

Jesnita wants to buy a medical insurance policy for herself. She compares the benefits of the policy plans from two different insurance companies, XX and YY.

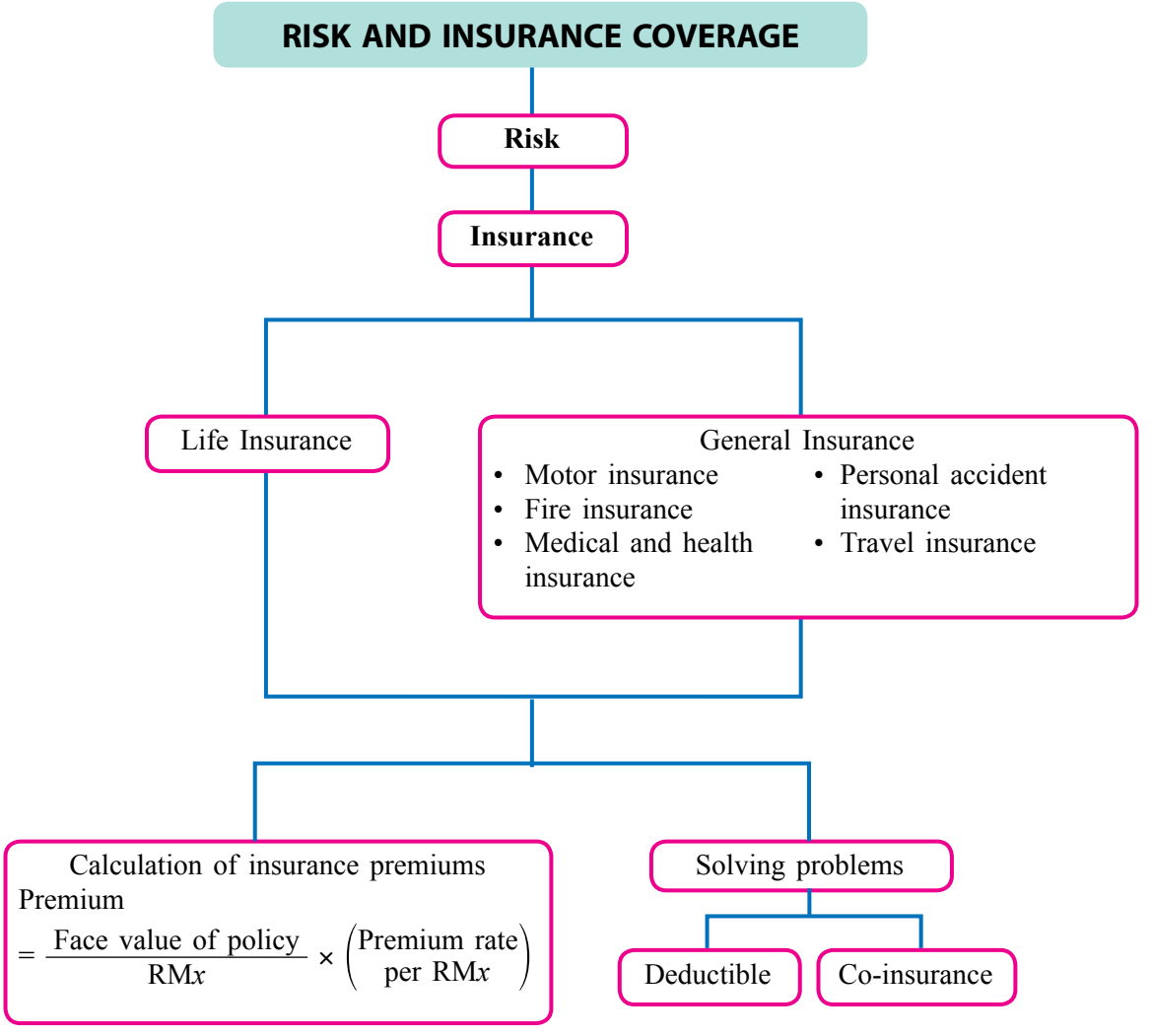
Benefit \ Insurance company	XX (RM)	YY (RM)
Overall annual limit	50 000	50 000
Hospital room and board	160 (maximum 365 days per annum)	200 (maximum 200 days per annum)
Intensive care unit	As charged (maximum 90 days per annum)	400 (maximum 90 days per annum)
Daily government hospital cash allowance	100 (maximum 365 days per annum)	50 (maximum 200 days per annum)
Annual premium	506.02	637.02

Based on the table, which insurance policy plan is better for Jesnita? Justify your answer.

Solution:

The XX policy plan is better because even though both provide the same overall annual limit of RM50 000, the annual premium for XX is lower than YY. Besides, the benefits offered are better in terms of coverage period, which is whole year-round for hospital room and board, and daily government hospital cash allowance. In addition, there is no coverage limit for intensive care unit because the compensation is paid according to the amount charged to the policyholder, unlike YY which has a limit of RM400 for the same coverage period.

Summary Arena



Reflection

At the end of this chapter, I can

explain the meaning of risk and the importance of insurance coverage, hence determine the types of life insurance and general insurance for protecting a variety of risks.		
investigate, interpret and perform calculations involving insurance rates and premiums.		
solve problems involving insurance including deductible and co-insurance.		

MINI PROJECT

Information about Encik Farhan:
Occupation: Police officer
Net salary per month: RM4 300
Monthly gross expenses: RM3 000
Monthly savings: RM500
Number of dependents: 3
Health status: Good

The above information is about Encik Farhan who has worked as a police officer for 7 years. He wants to buy life insurance that protects him and his family members. In your group, obtain information on life insurance policies from the internet. Select the best life insurance policy for Encik Farhan and his family members. Give your reason and present to the class.

Extensive Practice

Scan the QR code or visit bit.do/QuizE03 for interactive quiz

UNDERSTAND

1. Give three examples of insurable risks faced by a retailer who owns a shop.
2. Why is the principle of indemnity important in an insurance contract?

3. What is the difference between personal accident insurance, life insurance, and medical and health insurance?

4. An insurance company pays compensation for the motor insurance claim of a policyholder. The claim includes third-party damage and fire suffered by the policyholder's car. State the possible motor insurance policy purchased by the policyholder.

MASTERY

5. Vanitha bought a car from her friend without buying new motor insurance for the car because the existing motor insurance has not expired. She also did not contact the Road Transport Department Malaysia (JPJ) to change the ownership of the car to her name. One day, she was involved in an accident and the front part of her car was damaged.

   Based on the situation, can Vanitha make a claim from the insurance company that insured the car? Give your justification.

6. *Syarikat Insurans* ABC offers two life insurance plans with annual premium rates per RM1 000 face value as shown in the table on the right. The premium rates are for healthy and non-smoking men.

Plan / Age	35 years old	45 years old
5-year term	RM5.24	RM7.96
Yearly Renewable Term	RM4.03	RM5.52

   (a) Rizqi is 35 years old, a healthy and non-smoking man. He wants to buy a yearly renewable term plan of RM150 000. Calculate the monthly premium he has to pay.

   (b) (i) Vincent wants to buy a 5-year term with a face value of RM100 000. He is 45 years old, a healthy and non-smoking man. Calculate his annual premium.

   (ii) If Vincent buys a yearly renewable term plan with a face value of RM100 000, what is the difference in his annual premium, compared to the 5-year term plan?

7. Salleh's motorcycle is covered with a motor insurance with a deductible of RM150. During the insurance coverage period, Salleh suffered three accidents with losses of RM90, RM240 and RM300. State whether Salleh could claim for the losses. If yes, state the amount of compensation that can be claimed for each loss he suffered.

CHALLENGE

8. In 2019, Madam Karen was diagnosed with cancer and underwent surgery to treat her illness. The total surgery and treatment cost was RM130 000. In the following year, Madam Karen had an accident that fractured her arm. The treatment cost for her arm was RM12 000. If Madam Karen has a medical insurance policy within the two years with a deductible of RM5 000 per year and the annual limit of RM150 000, state the amount borne by Madam Karen and the amount of compensation paid by the insurance company for each treatment received in the two years.

9. Mr Gailang wants to buy fire insurance for his house. The insurance company estimates that the house's insurable value is RM1.3 million. The fire insurance policy that he wants to buy has a co-insurance provision of 70% of his property's insurable value and a deductible of RM3 000.

(a) Calculate the amount of required insurance for Mr Gailang.

(b) Mr Gailang's house caught on fire and his amount of losses was RM205 000. Calculate the amount of insurance purchased by Mr Gailang if he received a compensation of

(i) RM202 000

(ii) RM120 000

10. Encik Adam's house is insured with fire insurance with a co-insurance provision of 80% of his property's insurable value. The house's insurable value is RM250 000.

(a) Calculate the amount of required insurance for Encik Adam.

(b) Encik Adam has insured his house according to the amount in (a). If his house is completely burnt down, will he receive the compensation amount of RM250 000? Give your justification.

11. Sofia's medical cost covered by her insurance policy is RM10 800. She wants to claim compensation from her major medical insurance policy with a deductible allocation of RM300 and a co-insurance percentage participation of 80/20. Calculate the amount borne by Sofia and the amount of compensation she received.

EXPLORING MATHEMATICS

According to the data from Bank Negara Malaysia, only 41 percent of Malaysians are covered with at least one insurance policy at the moment.

Source: Berita Harian, 29 July 2019, Government targets national insurance penetration rate to breach 75 percent in 2020

During Mathematics Week at school, your group needs to give a speech on the lack of awareness of having an insurance policy among Malaysians. Your speech must include the following aspects:

✓ The factors affecting a lack of awereness towards having insurance among Malaysians

✓ The importance of having an insurance as a protection

Discuss this in your group and choose a presenter to give the speech in class.

CHAPTER 4

Consumer Mathematics: Taxation

What will you learn?

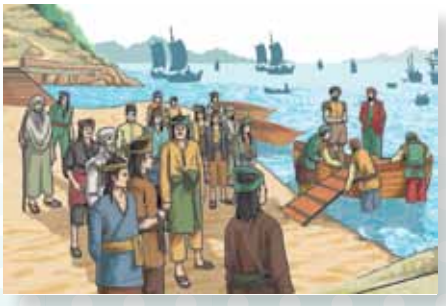
- Taxation

Why study this chapter?

Accounting is a field of study that involves taxation. Accountants require a comprehensive understanding of the economy and type of taxes in preparing corporate or personal income tax statements.

Do you know?

The Malaccan Sultanate had a well-organised tax system that was imposed on merchants based on their relationship with Malacca. The first income tax system was introduced in the Federation of Malaya in 1947 under the Income Tax Ordinance 1947. It is subsequently superseded by the Income Tax Act 1967.



For more information:

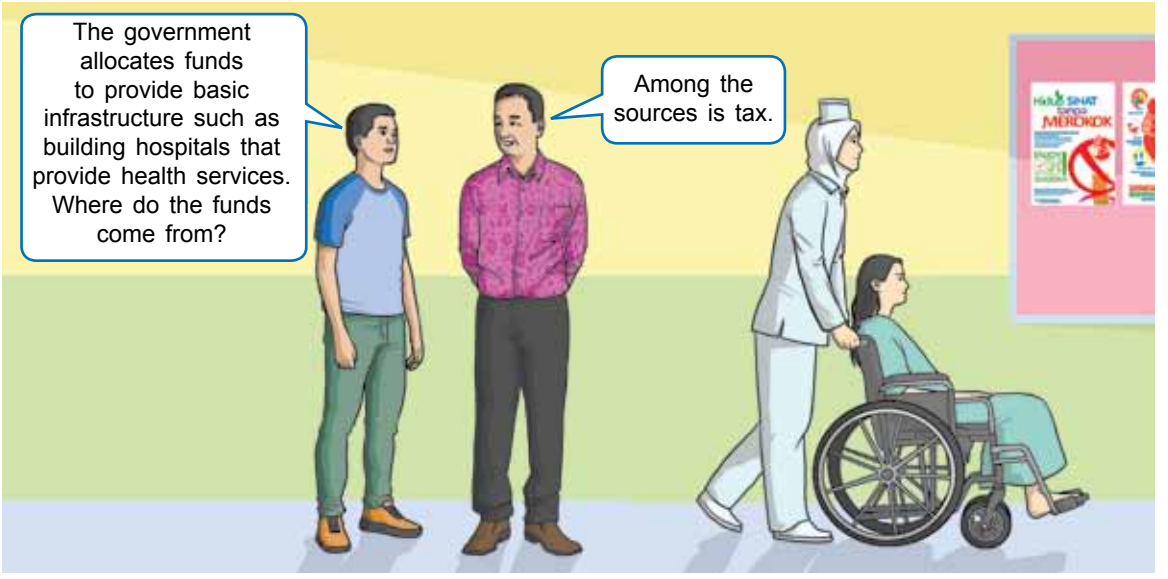
bit.do/DoYouKnowChap4

WORD BANK

road tax	*cukai jalan*
sales and service tax	*cukai jualan dan perkhidmatan*
income tax	*cukai pendapatan*
property assessment tax	*cukai pintu*
quit rent	*cukai tanah*
tax relief	*pelepasan cukai*
monthly tax deduction	*potongan cukai bulanan*
rebate	*rebat*
separate tax assessment	*taksiran cukai berasingan*
joint tax assessment	*taksiran cukai bersama*

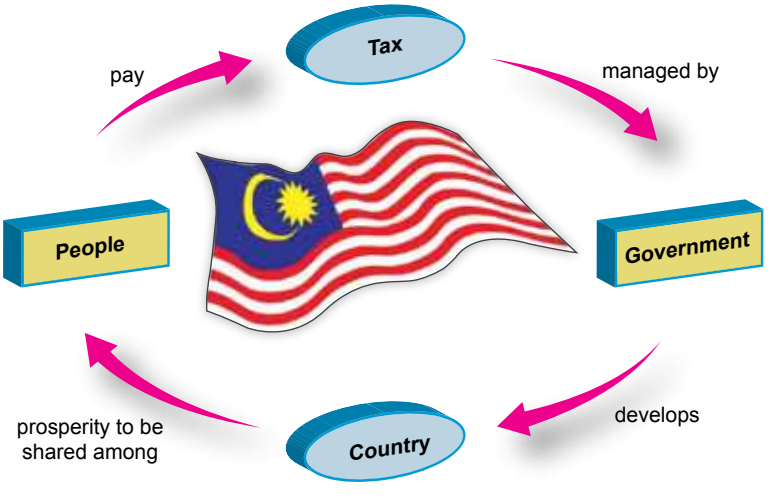
Governing a country is a complicated process and requires huge funding. Taxes are the main source of income for a government to manage and develop a country. An optimal tax system not only serves as a source of revenue for the government but also improves the quality of life of its citizens and encourages more private investments. What are the taxes in Malaysia?

4.1 Taxation



What is the purpose of taxation?

Taxation is a process of revenue (money) collection from individuals or companies, for use in the country's development, by providing various facilities (for example, education, healthcare, national security, welfare) for the well-being of all citizens. All taxes collected are based on parliament-approved acts.



Learning Standard

Explain the purpose of taxation.

My Malaysia

Inland Revenue Board (IRB) and Royal Malaysian Customs Department are responsible for direct and indirect tax collection in Malaysia. The state government is responsible for the collection of revenue from mining, land and forests, revenue from the local authority and acquisition from the issuance of licenses except those collected by the Federal Government.

MIND MOBILISATION 1 Group

Aim: To explain the purpose of taxation.

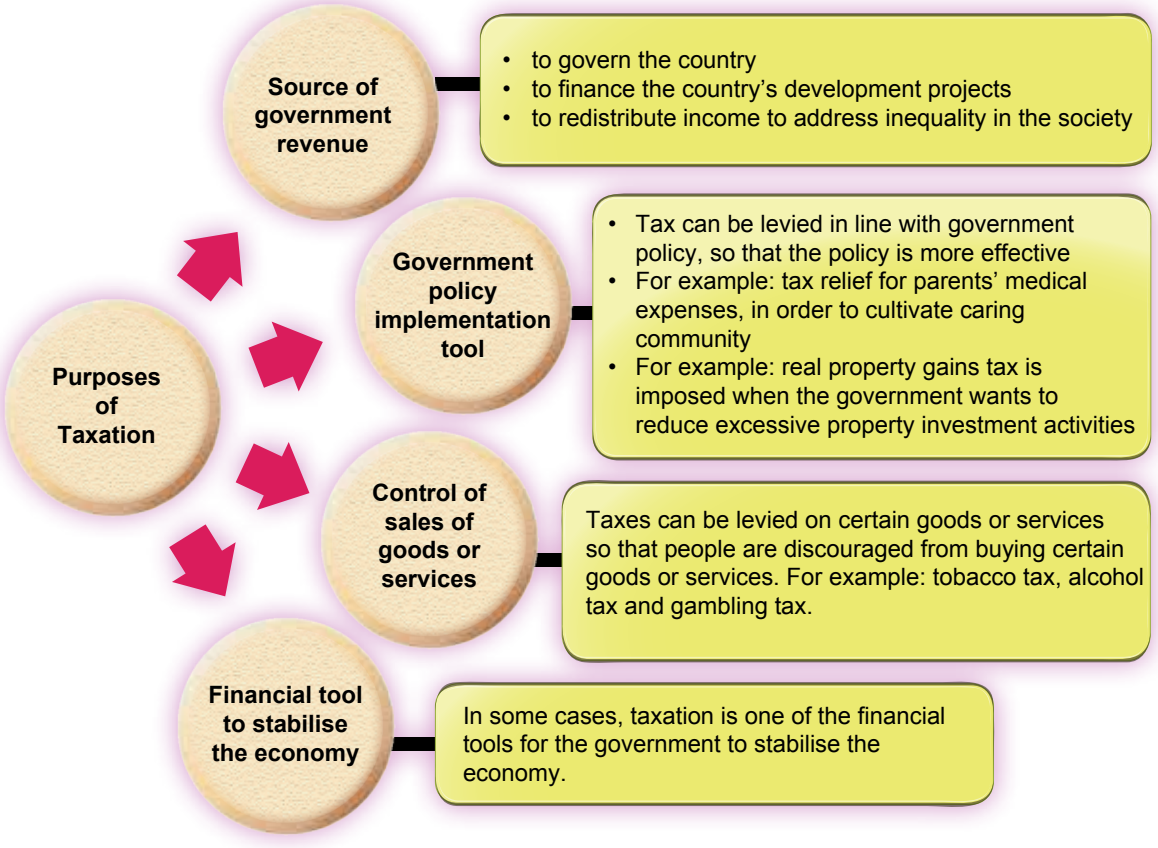
Steps:

1. Divide the class into groups of 4 pupils.
2. Each group browses the internet for further information that relates to the following questions.
   (a) What is the purpose of taxation?
   (b) Describe 5 examples of how the government uses tax.
3. The information collected is presented in an appropriate mind map on a presentation paper and to be pasted on the wall of the classroom.
4. Pupils comment on the work of other groups by attaching sticky notes on the presentation paper.

Discussion:

1. Explain the purpose of taxation and example of tax usage by the government.
2. Next, discuss the responsibilities and attitudes of a taxpayer who has good moral and ethics.

The purposes of taxation can be summarised as in the diagram below.



Tax payment is a compulsory responsibility for each person. The citizens should pay their tax honestly in compliance with the laws so that the government does not suffer losses. National development planning will be affected if the country loses its tax revenue due to dishonest and irresponsible taxpayers.

Interactive Platform

How can the government utilise tax to stabilise the economy?

Self Practice 4.1a

1. What are the purposes of taxation?

2. **RM91.8 million RPGT collected as of 9 July**

   KUALA LUMPUR: The government raised RM91.868 million through the Real Property Gains Tax (RPGT) as of 9 July for property disposal activities for the holding period exceeding five years.

   Source: Berita Harian, 17 July 2019

   Real Property Gains Tax (RPGT) is levied on any profit earned through the sales of land or real property where the resale price is higher than the purchase price. What is the purpose of this tax? Explain your answer.

Interactive Platform

Siti works as a clerk in a company for 5 years. She thinks that her pay is low and will not be taxable. She has never reported and paid any income tax all these times.

Is Siti a responsible citizen? Explain your answer.

What are the types of taxes and the consequences of tax evasion from legal and financial aspects?

Learning Standard

Describe various taxes, hence the consequences of tax evasion from legal and financial aspects.

Income Tax

- The tax imposed on income earned by a salaried individual or a company operating in Malaysia.
- Each salaried individual and profitable company is required to register as a taxpayer, assess annual income and pay income tax annually.
- Report all income, tax exemption, tax reduction, tax relief and tax rebate by submitting Income Tax Return Form, ITRF (in Malay *Borang Nyata Cukai Pendapatan, BNCP*), either by form, or electronically via internet (*ezHasil*).
- **Self-Assessment System** is implemented. Taxpayers should make self-assessment, keep records and pay a certain amount of tax.

1. Self-Assessment (making own income tax calculation)

- Assessment is made for the previous year, for example assessment for year 2019 is made in year 2020.
- Deadline for submission of ITRF and tax payment for an individual who does not own a business (BE form) is 30 April, while for an individual who owns a business (B form) is 30 June.

2. Keep Records

- All documents such as forms and original copy of receipts must be kept for 7 years.

3. Payment

- Tax payment can be made in a lump sum to the Inland Revenue Board (IRB) or through monthly tax deduction (PCB).

Who is collecting?

- Inland Revenue Board (IRB), or in Malay *Lembaga Hasil Dalam Negeri* (*LHDN*).

Consequences of tax evasion

Tax Evasion	Penalty	Legal act
Failure to submit annual report if eligible for tax payment	Fine of RM200 up to RM20 000 or imprisonment of not exceeding 6 months or both	Income Tax Act 1967 (Act 53) Section 112(1)
Make omission on any income in ITRF	Fine of RM1 000 up to RM10 000 and penalty of 200% of the amount of tax which has been undercharged	Income Tax Act 1967 (Act 53) Section 113(1)(a)
Give incorrect information that affects income tax payable for own or other person	Fine of RM1 000 up to RM10 000 and penalty of 200% of the amount of tax which has been undercharged	Income Tax Act 1967 (Act 53) Section 113(1)(b)
Evade intentionally or assist other person to evade tax	Fine of RM1 000 up to RM20 000 or imprisonment of not exceeding 3 years or both and penalty of 300% of the amount of tax which has been undercharged	Income Tax Act 1967 (Act 53) Section 114(1)

* Subject to change from time to time.
(Source: bit.do/AktaCukaiPendapatan) (Accessed on 14 September 2020)

Road Tax

Info Bulletin

Example of *Lesen Kenderaan Motor (LKM)*

- The tax levied on road user who owns vehicles including motorcycle and car.
- Vehicle owner must pay road tax before the existing road tax expires. The expiry date is printed on road tax receipt (*Lesen Kenderaan Motor*) that must be affixed or shown on the vehicle.
- Vehicle with expired road tax of more than one year must be sent to Puspakom for inspection. Puspakom inspection report and tax payment must be submitted to Road Transport Department in order to reactivate the road tax.
- If the road tax is not paid, the vehicle will not be allowed on the road.

Who is collecting?

- Road Transport Department, or in Malay *Jabatan Pengangkutan Jalan* (*JPJ*).

Consequences of tax evasion

Tax evasion	Penalty	Legal act
Continually using a vehicle without valid road tax	Fine of not exceeding RM2 000	Road Transport Act 1987 (Act 333) Section 23(1)

* Subject to change from time to time.
(Source: bit.do/AktaPengangkutanJalan) (Accessed on 14 September 2020)

Property Assessment Tax

- The tax levied on all holdings or properties (residential houses, industrials, commercial buildings and vacant lands) in order to cover municipal expenses such as rubbish collection, drain cleaning and maintaining public parks.

Who is collecting?

- Local authority, such as municipal council or district council.

Consequences of tax evasion

Tax Evasion	Action	Legal Act
Does not pay assessment tax within the designated period	Notice in Form E will be issued	Local Government Act 1976 (Act 171) Section 147 and 148
Does not pay assessment tax within 15 days after the notice issued	Detention warrant in Form F will be issued and portable property could be seized	Local Government Act 1976 (Act 171) Section 147 and 148

Info Bulletin

Detention warrant is a power of attorney which enables local authorities to seize properties in a building through the Registrar of the High Court in accordance with Section 151.

* Subject to change from time to time.
(Source: bit.do/AktaKerajaanTempatan) (Accessed on 14 September 2020)

Quit Rent

- The tax levied on the owner of agricultural land, corporate land and land with building.
- This tax is interpreted as Lease under Section 5 of National Land Code 1965.

Who is collecting?

- State land authority, which is State Land Office.

Info Bulletin

Parcel tax is a tax levied on stratified properties (for example, apartments and condominiums) to replace quit rent.

Consequences of tax evasion

Tax evasion	Action	Legal act
Does not pay quit rent within the designated period	Notice of claim in Form 6A will be issued	National Land Code 1965 Section 97
Does not pay quit rent within the stated period in the notice of claim	The land can be seized	National Land Code 1965 Section 100

* Subject to change from time to time.
(Source: tanahweb.wordpress.com/ktn/) (Accessed on 14 September 2020)

Sales and Service Tax

- Sales and Service Tax enforced on 1 September 2018, consists of two parts, namely sales tax and service tax.
- Sales tax is a tax levied only once on various taxable goods at the stage of manufacturing or during importation.
- Service tax is a tax levied on consumer who receives taxable services, such as hotel services, insurance and *takaful*, food and beverages preparation, telecommunications and credit cards.
- Manufacturers or importers with sales value of taxable goods exceeding RM500 000 per annum shall register under Sales Tax Act 2018.
- Service providers with value of taxable services exceeding threshold of RM500 000 per annum as well as food and beverage service providers with value of taxable services exceeding threshold of RM1 500 000 per annum, shall register under Service Tax Act 2018.
- Tax collection is a responsibility of service providers on the government's behalf. Service tax must be collected from the customer who receives the services. Service tax collected by the service providers will be paid to the government within a designated period.

Info Bulletin

Sales and Service Tax is not imposed in tax-free territories, for example Langkawi Island and Pengkalan Kubor, Kelantan.

* Subject to change

Scan the QR code or visit the following websites:

list of taxable goods

bit.do/TaxableGoods

list of taxable services

bit.do/TaxableService

Info Bulletin

Threshold is the annual sales turnover of a business which determines the liability to be registered under Sales and Service Tax Act.

(Source: bit.do/Threshold)

(Accessed on 17 July 2020)

Who is collecting?

- Royal Malaysian Customs Department (RMCD), or in Malay *Jabatan Kastam Diraja Malaysia (JKDM)*.

Consequences of tax evasion

Tax evasion	Penalty	Legal act
Evade tax intentionally or assist other person to evade tax	First offence: fine of 10 times up to 20 times the amount of sales tax or imprisonment of not exceeding 5 years or both	Sales Tax Act 2018 (Act 806) Section 86(1) and 86(2)
	Second or subsequent offence: fine of 20 times up to 40 times the amount of sales tax or imprisonment of not exceeding 7 years or both	
The amount of tax evasion that cannot be ascertained	Fine of RM50 000 up to RM500 000 or imprisonment of not exceeding 7 years or both	Sales Tax Act 2018 (Act 806) Section 86(3)
The party who assist in preparation of false tax statements	Fine of RM2 000 up to RM20 000 or imprisonment of not exceeding 3 years or both	Sales Tax Act 2018 (Act 806) Section 86(4)
Similar penalties for tax evasion also stipulated in Service Tax Act 2018 (Act 807) Section 71		

* Subject to change from time to time.
(Source: bit.do/AktaCukaiJualan)
(Source: bit.do/AktaCukaiPerkhidmatan)
(Accessed on 14 September 2020)

Self Practice 4.1b

1. Explain the meaning of property assessment tax.

2. Mr Tan owns a car. The road tax has expired on 7 June 2019. He forgot to renew the road tax.

Describe the consequences if Mr Tan is still driving the car.

3. Describe the parties who should register under Sales Tax Act 2018.

4. What are the differences between quit rent and property assessment tax?

Quit rent	Property assessment tax

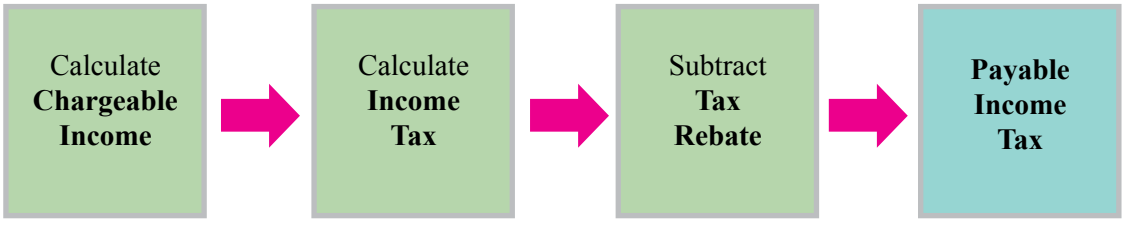
5. What are the consequences if someone deliberately evades income tax?

How to calculate income tax?

Learning Standard

Investigate, interpret and perform calculations involving various taxes.

Income tax to be paid by an individual can be calculated through the calculation process as shown in the flow chart below.



Chargeable Income

Individual income tax to be paid is calculated based on the amount of **chargeable income**. Chargeable income can be obtained by using the equation below.

Chargeable income
= total annual income – tax exemption – tax relief

Info Bulletin

Income tax is a type of progressive tax, in which tax rates increase as income increases.

(A) Total Annual Income

Total annual income includes all forms of wages, salaries, bonuses, dividends, interest, rentals, royalties, premiums and so on. However, not all the income received from the employer is taxable, for example, petrol or transport allowance, employer's subsidies such as housing and car subsidies, medical benefits, *Elaun Bantuan Sara Hidup* (COLA) and *keraian*.

Info Bulletin

Example of statement of Remuneration from Employment

bit.do/StatementOf Remuneration

(B) Tax Exemption

Tax exemption refers to all personal expenses in the forms of gifts, donations, contributions to government or various organisations, for example state libraries and healthcare organisations.

These expenses may be tax exempted but are subjected to certain limits in amount. These tax exemptions aim to promote developments in cultural and arts as well as in cultivating a caring society.

Info Bulletin

From 2015 onwards, an individual who earns an annual employment income of RM34 000 (after EPF deduction) has to register a tax file. (*Source: Inland Revenue Board of Malaysia*)

(C) Tax Relief

Tax reliefs refer to items or expenses, which are not taxed, for the benefit of the individual or family members, for example Employees' Provident Fund (EPF), medical treatment and education fees. Items of tax relief depend on the policy set by the government and may change from year to year.

Scan the QR code or visit bit.do/TaxReliefs for a complete list of income tax reliefs.

Income Tax

After obtaining chargeable income, income tax can be determined by referring to tax rates table. **Tax rates** are fixed by the government and may vary based on the government policy on the annual budget. The table below is individual income tax rates for Assessment Year of 2020.

Table: Individual Income Tax Rates for Assessment Year of 2020

Chargeable Income (RM)	Calculations (RM)	Rate (%)	Tax (RM)
0 - 5 000	On the first 5 000	0	0
5 001 - 20 000	On the first 5 000 Next 15 000	 1	0 150
20 001 - 35 000	On the first 20 000 Next 15 000	 3	150 450
35 001 - 50 000	On the first 35 000 Next 15 000	 8	600 1 200
50 001 - 70 000	On the first 50 000 Next 20 000	 14	1 800 2 800
70 001 - 100 000	On the first 70 000 Next 30 000	 21	4 600 6 300
100 001 - 250 000	On the first 100 000 Next 150 000	 24	10 900 36 000
250 001 - 400 000	On the first 250 000 Next 150 000	 24.5	46 900 36 750
400 001 - 600 000	On the first 400 000 Next 200 000	 25	83 650 50 000
600 001 - 1 000 000	On the first 600 000 Next 400 000	 26	133 650 104 000
1 000 001 - 2 000 000	On the first 1 000 000 Next 1 000 000	 28	237 650 280 000
Exceeding 2 000 000	On the first 2 000 000 Next every ringgit	 30	517 650 …

* Subject to change from time to time.
(*Source: Official Portal Inland Revenue Board of Malaysia*)

Tax Rebate

Tax rebate is given to reduce tax to be paid. There are 2 types of tax rebate, namely:

(a) Tax rebate of RM400 will be given to taxpayer if the chargeable income does not exceed RM35 000, subject to change.

(b) The amount of *zakat* or *fitrah* paid by Muslim citizens.

Info Bulletin

Zakat means clean, holy or blessed. *Zakat* is a donation of a propotion of wealth to be distributed to certain groups of people (such as the poor), in accordance to certain conditions.

Example 1

Encik Khairul has an annual income of RM125 300, including allowances. It is given that the allowances amounting to RM12 340 are tax exempted. At the same time, he has made a donation to a library amounting to RM2 000. The total tax relief, on the other hand, is RM22 500. Calculate Encik Khairul's chargeable income.

Solution:

Total annual income
= RM125 300 – RM12 340
= RM112 960

← Annual income should be deducted by tax exempted allowances

Chargeable income
= total annual income – tax exemption – tax relief
= RM112 960 – RM2 000 – RM22 500
= RM88 460

← Tax exemption includes donation to library amounting to RM2 000

Example 2

Asraf had a chargeable income of RM47 531 in 2020. He had paid *zakat* amounting to RM280 in that year. Calculate the income tax to be paid by Asraf for that year.

Solution:

Asraf's chargeable income, RM47 531 lies in the range of chargeable income as below.

Chargeable Income (RM)	Calculations (RM)	Rate (%)	Tax (RM)
35 001 - 50 000	On the first 35 000 Next 15 000	 8	600 1 200

Tax on the first RM35 000 = RM600 ← Base tax

Tax on the next balance = (RM47 531 – RM35 000) × 8%
= RM1 002.48

Eligible rebate (*zakat*) = RM280

Income tax = RM600 + RM1 002.48 – RM280
= RM1 322.48

← Income tax
= base tax + tax on the next balance – tax rebate

Info Bulletin

Income tax rate for company for Assessment Year of 2019:

Company with paid up capital not more than RM2.5 million. (i) On first RM500 000 (ii) Subsequent balance	 17% 24%
Company with paid up capital more than RM2.5 million.	24%

CHAPTER 4

Example 3

Mr Chan had a total annual income of RM52 770 in the year 2020. He donated RM300 to a government-approved charity organisation. The table below shows the tax reliefs claimed by Mr Chan.

Tax relief	Amount (RM)
Individual	9 000
Life insurance and EPF (limited to RM7 000)	5 800
Medical insurance (limited to RM3 000)	1 700
Self education fees (limited to RM7 000)	4 000

Calculate the income tax payable by Mr Chan for the particular year, by taking into account the tax rebates, if eligible.

Info Bulletin

An individual does not have to make tax assessment in the first year of employment because he or she does not have to pay income tax. However, he or she is advised to register the IRB income tax file as he or she may experience a pay increase in the coming years.

Solution:

Chargeable income
= total annual income – tax exemption – tax relief
= RM52 770 – RM300 – (RM9 000 + RM5 800 + RM1 700 + RM4 000)
= RM31 970

Interactive Platform

What is the difference between tax relief and tax rebate?

Mr Chan's chargeable income, RM31 970 lies in the range of chargeable income as below.

Chargeable Income (RM)	Calculations (RM)	Rate (%)	Tax (RM)
20 001 - 35 000	On the first 20 000 Next 15 000	 3	150 450

Tax on the first RM20 000 = RM150

Tax on the next balance = (RM31 970 – RM20 000) × 3%
= RM359.10

Eligible rebate = RM400 ← Rebate of RM400 due to chargeable income ⩽ RM35 000

Income tax payable = RM150 + RM359.10 – RM400
= RM109.10

What is monthly tax deduction (*Potongan Cukai Bulanan, PCB*) and related calculation?

Monthly Tax Deduction (*Potongan Cukai Bulanan, PCB*) is salary deduction of an employee for income tax payment of current year. The deduction amount is based on the table of monthly tax deduction or *PCB* computational calculation method provided by Inland Revenue Board (IRB).

Info Bulletin

Minimum monthly salary eligible for *PCB* (after EPF deduction) for single individual is RM2 851 and for married individual whose spouse is not working is RM3 851.

*Subject to change

Taxpayers should make comparison between total amount of *PCB* deducted and tax payable as shown below.

- Tax payable > *PCB*
  Tax – *PCB* = Insufficient tax payment
  Insufficient tax payment must be made to IRB
- Tax payable < *PCB*
  PCB – Tax = Excess deduction
  Excess deduction of *PCB* will be refunded by IRB to taxpayer's bank account

Example 4

Mr Lim's annual salary was RM74 000 in 2020. He claimed the following tax reliefs: individual for RM9 000, life insurance and EPF for RM7 000, medical insurance for RM1 325 and medical treatment expenses for his mother for RM1 250. He has also donated RM1 000 to an approved welfare centre. His salary was deducted monthly by RM180 for monthly tax deduction (*PCB*).

(a) Calculate the income tax payable for that year.

(b) Does Mr Lee need to pay any more income tax after the monthly deductions? Explain your answer.

Solution:

(a) Chargeable income = total income – tax exemption – tax relief
= RM74 000 – RM1 000 – (RM9 000 + RM7 000 + RM1 325 + RM1 250)
= RM54 425

Mr Lim's chargeable income, RM54 425 lies in the range of chargeable income as below.

Chargeable Income (RM)	Calculations (RM)	Rate (%)	Tax (RM)
50 001 - 70 000	On the first 50 000 Next 20 000	 14	1 800 2 800

Tax on the first RM50 000 = RM1 800
Tax on the next balance = (RM54 425 – RM50 000) × 14%
= RM619.50

Eligible rebate = RM0 ← Rebate of RM0 due to chargeable income > RM35 000

Income tax payable = RM1 800 + RM619.50 – RM0
= RM2 419.50

(b) Total *PCB* deducted = RM180 × 12
= RM2 160 ← less than tax payable

Tax payable > *PCB*. Thus, there is insufficient tax payment.

Insufficient tax payment = RM2 419.50 – RM2 160
= RM259.50

Mr Lim needed to pay RM259.50 to IRB because the deducted *PCB* was insufficient for his income tax.

What is joint tax assessment and separate tax assessment and related calculation?

For married individuals, income tax assessment system allows assessment to be done based on either one of the following assessments:

Joint tax assessment
Tax assessment filed by husband (or wife). Wife's income (or husband's income) is aggregated with husband's (or wife's) when filing tax assessment.

Separate tax assessment
Husband and wife each file their own tax assessment.

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Terms for filing joint tax assessment:
(a) Husband and wife must live together
(b) Has total income to be aggregated with the total income of the spouse
(c) The husband / wife who elects for joint assessment must be a Malaysian citizen, if not a resident in Malaysia
(d) Can only be done with one wife

In general, if both husband and wife have separate income, joint tax assessment may reduce the total amount of tax reliefs. This is because most of the tax reliefs have a claim limit, irrespective of whether it is separate or joint tax assessment. This will cause the tax payment to increase dramatically.

Example 5

Mr Rajan and his wife received annual salaries of RM53 000 and RM57 000 respectively in 2020. They each donated RM500 to a government-approved welfare organisation in that year. The following are the tax reliefs to be claimed by Mr Rajan and his wife.

Item	Mr Rajan	Wife
Individual	RM9 000	RM9 000
Lifestyle (limited to RM2 500)	RM2 500	RM2 700
Life insurance (limited to RM7 000)	RM5 830	RM6 000
Medical insurance (limited to RM3 000)	RM2 650	RM2 570

(a) Calculate the total income tax of Mr Rajan and his wife by using joint tax assessment and separate tax assessment.

(b) Between joint tax assessment and separate tax assessment, which is more suitable to be used by Mr Rajan and his wife? Explain your answer.

Solution:

(a) The table below shows the calculation of income tax for Mr Rajan and his wife using joint tax assessment and separate tax assessment.

Item	Joint tax assessment	Separate tax assessment	
	Husband and wife	Husband	Wife
Total income	RM53 000 + RM57 000 = RM110 000	RM 53 000	RM 57 000
Total exemption (Donation)	– RM 1 000	– RM 500	– RM 500
Tax relief			
– Individual	– RM 9 000	– RM 9 000	– RM 9 000
– Lifestyle (limited to RM2 500)	– RM 2 500	– RM 2 500	– RM 2 500 The tax relief that is eligible to be claimed is RM2 500 despite spending RM2 700
– Life insurance (limited to RM7 000)	– RM 7 000	– RM 5 830	– RM 6 000
– Medical insurance (limited to RM3 000)	– RM 3 000	– RM 2 650	– RM 2 570
Chargeable income	RM 87 500	RM 32 520	RM 36 430
Base tax	RM 4 600	RM 150	RM 600
Tax on the next balance	Balance = 87 500 – 70 000 = 17 500 RM17 500 × 21% = RM3 675.00	Balance = 32 520 – 20 000 = 12 520 RM12 520 × 3% = RM375.60	Balance = 36 430 – 35 000 = 1 430 RM1 430 × 8% = RM114.40
Tax rebate	– RM 0	– RM 400 (Chargeable income ⩽ RM35 000)	– RM 0
Income tax payable	RM4 600 + RM3 675 – RM0 = **RM8 275**	RM150 + RM375.60 – RM400 = RM125.60	RM600 + RM114.40 – RM0 = RM714.40
		RM840	

(b) Separate tax assessment is more suitable to be used because income tax payable is far lower, which is RM840, as compared to RM8 275 for joint tax assessment.

Smart TiPS

In joint tax assessment, the claimable tax relief is considered for calculation only once.

How to calculate road tax?

Road tax is levied based on the engine capacity of the vehicle used by the owner. Normally, road tax will be higher for a vehicle with a higher engine capacity.

Scan the QR code or visit bit.do/RoadTax for road tax rates in Malaysia.

Example 6

The table below shows the road tax rates for motorcycle in Sabah and Sarawak.

Engine Capacity	Road Tax Rate	
	Base Rate	Progressive Rate
151 cc - 200 cc	RM 9.00	-
201 cc - 250 cc	RM 12.00	-
251 cc - 500 cc	RM 30.00	-

Mr Chong owns a motorcycle, with an engine capacity of 260 cc, for private use in Sarawak. Calculate the road tax for his motorcycle.

Solution:

Road tax for motorcycle (260 cc) = RM30

Example 7

The table below shows the road tax rates for private car in Peninsular Malaysia.

Engine Capacity	Road Tax Rate	
	Base Rate	Progressive Rate
1 000 cc and below	RM 20.00	-
1 001 cc - 1 200 cc	RM 55.00	-
1 201 cc - 1 400 cc	RM 70.00	-
1 401 cc - 1 600 cc	RM 90.00	-
1 601 cc - 1 800 cc	RM 200.00	+ RM 0.40 each cc exceeding 1 600 cc
1 801 cc - 2 000 cc	RM 280.00	+ RM 0.50 each cc exceeding 1 800 cc

Alan has two private cars in Malacca, with engine capacity of 859 cc and 1 997 cc respectively. Calculate the road tax for both of his cars.

Solution:

Road tax for car (859 cc) = RM20.00

Road tax for car (1 997 cc)
= RM280.00 + (1 997 – 1 800) × RM0.50
= RM280.00 + 197 × RM0.50
= RM280.00 + RM98.50
= RM378.50

Car with engine capacity 1 801 cc up to 2 000 cc is levied RM280 base rate and RM0.50 progressive rate for each cc exceeding 1 800 cc

How to calculate property assessment tax?

Property assessment tax rate is set by local authority based on location and type of property. The amount of tax payable each year depends on the tax rate charged on the annual value.

Info Bulletin

Annual value
= estimated monthly rental
× 12 months

Property assessment tax = property assessment tax rate × annual value

Annual value (or **annual assessment**) is a reasonable gross estimation of annual rental that is expected to be obtained in a year from a property, if it is rented out.

Example 8

Mariani owns a residential house in Subang Jaya. She receives property assessment tax bill from Subang Jaya Municipal Council. It is given that the annual value is RM6 580 and the property assessment tax rate is 5%. Calculate the property assessment tax payable by Mariani for each half-year.

Solution:

Property assessment tax = property assessment tax rate × annual value
= 5% × RM6 580
= RM329 per year

$$\text{Property assessment tax for each half-year} = \frac{\text{RM329}}{2}$$
= RM164.50 each half-year

How to calculate quit rent?

Quit rent rate is fixed by the state government based on location, size and type of land. The amount of tax payable each year depends on the quit rent rate per unit area on the total land area owned.

Quit rent = quit rent rate per unit area × total land area

Example 9

Mr Hamid owns a house with an area of 130 m². It is given that the quit rent rate levied is RM0.43 per square metre. Calculate the quit rent payable by Mr Hamid each year.

Solution:

Quit rent = quit rent rate per unit area × total land area
= RM0.43 × 130
= RM55.90 each year

How to calculate sales and service tax?

Sales tax rates on goods are different, which is 5%, 10% or other rates depending on the goods, while **service tax** rate is 6%.

Info Bulletin

Service tax on each credit card is RM25 per year.

Example 10

Sofia rented a room at Hotel Selesa for RM240 per night. Sofia stayed at the hotel for two nights. It is given that the hotel charges 6% for service tax. Calculate the service tax to be paid by Sofia.

Solution:

Service tax = RM240 × 2 × 6%
= RM28.80

Self Practice 4.1c

1. Calculate the chargeable income based on the information given.

(a) Total income RM89 340

Tax exemption	RM 2 500
Tax relief	RM11 000

(b) Total income RM 76 000

Tax exemption	RM 8 250
Tax relief	RM 15 340

(c) Total income RM68 210

Tax relief	
Individual	RM9 000
Life insurance and EPF (limited to RM7 000)	RM6 250
Medical insurance (limited to RM3 000)	RM3 600

(d) Total income RM113 421

Tax relief	
Individual	RM 9 000
Life insurance and EPF (limited to RM7 000)	RM 11 420
Education insurance (limited to RM3 000)	RM 2 400

2. Calculate the income tax payable for each of the following by taking into consideration tax rebate, if eligible.

(a)

Chargeable income	RM108 630

(b)

Chargeable income	RM28 440

(c)

Chargeable income	RM35 270
Zakat	RM 500

(d)

Chargeable income	RM 52 394
Zakat	RM 1 250

3. Puan Aishah had a total annual income of RM61 593 in 2020. The total tax relief claimed by her was RM14 320. She paid *zakat* amounting to RM700 that year. Calculate the income tax payable by Puan Aishah.

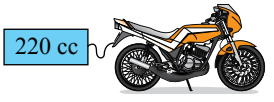
4. Puan Siew's chargeable income is RM75 311. Each month, RM300 is deducted from her salary for monthly tax deduction (*PCB*). However, the total *PCB* is insufficient for her income tax payment. How much is the income tax balance payable to IRB?

5. Mr Subra and his wife each earned an annual salary of RM61 000 and RM56 000 in 2020. The following are the expenses claimed by Mr Subra and his wife for that year.

Item	Mr Subra	Wife
Individual	RM9 000	RM9 000
Life insurance and EPF (limited to RM7 000)	RM6 000	RM5 900
Lifestyle (limited to RM2 500)	RM2 000	RM2 500
Net deposit in SSPN (limited to RM8 000)	RM5 000	RM6 000
Education insurance (limited to RM3 000)	RM2 600	RM2 000

Calculate the total income tax for Mr Subra and his wife using separate tax assessment.

6. Firdaus has a motorcycle and a car with the engine capacity as shown in the diagram below. Based on the table below, calculate the road tax for both types of vehicles.



Engine Capacity	Road Tax Rate	
	Base Rate	Progressive Rate
Motorcycle Road Tax Rate		
150 cc and below	Free	-
151 cc - 200 cc	RM 30.00	-
201 cc - 250 cc	RM 50.00	-
Private Car Road Tax Rate		
1 401 cc - 1 600 cc	RM 90.00	-
1 601 cc - 1 800 cc	RM 200.00	+ RM 0.40 each cc exceeding 1 600 cc
1 801 cc - 2 000 cc	RM 280.00	+ RM 0.50 each cc exceeding 1 800 cc
2 001 cc - 2 500 cc	RM 380.00	+ RM 1.00 each cc exceeding 2 000 cc

7. Madam Kim owns a service apartment in Shah Alam. The property assessment tax is 5%. It is estimated that the house rental is at RM1 500 per month. Calculate the property assessment tax payable by Madam Kim each year.

8. Encik Abdullah owns a 300 m² residential land in Kedah. It is given that the quit rent rate is RM0.15 per square metre. Calculate the quit rent payable by Encik Abdullah each year.

9. Yi Soon had dinner at a restaurant. He had a plate of fried rice, two pieces of fried chicken and a cup of coffee. The table below shows the price of the food and drink.

Food and Drink	Price
Fried rice	RM6.00 / plate
Fried chicken	RM5.00 / piece
Coffee	RM4.00 / cup

It is given that the restaurant charges 6% on service tax. Calculate Yi Soon's bill.

How to solve problems involving taxation?

Example 11

Learning Standard

Solve problems involving taxation.

Sabrina has a chargeable income of RM30 633 in 2020. She has paid *zakat* amounting to RM230 in that year.
(a) How much is Sabrina's tax rebate?
(b) Calculate the income tax payable by Sabrina.

Solution:

Understanding the problem

Chargeable income
= RM30 633
Rebate (*zakat*) = RM230

Devising a strategy

(a) Determine rebate for *zakat* and rebate for chargeable income ⩽ RM35 000.
(b) Calculate the income tax by referring to tax rate table and total rebate.

Implementing the strategy

(a) Sabrina's total tax rebate = RM400 + RM230
= RM630

← Rebate of RM400 for chargeable income ⩽ RM35 000 and rebate of RM230 for *zakat* payment

(b) Tax on the first RM20 000 = RM150

Tax on the next balance = (RM30 633 – RM20 000) × 3%
= RM10 633 × 3%
= RM318.99

Income tax payable
= RM150 + RM318.99 – RM630
= (–RM161.01)
= RM0

← Excess rebate will not be refunded to taxpayer. Thus, tax payable = RM0

Making a conclusion

(a) RM630 (b) RM0

Example 12

The table below shows the tax exemption and tax relief items for Encik Hanafi.

Individual	RM9 000
Life insurance and EPF (limited to RM7 000)	RM6 000
Donation to library	RM 500

It is given that Encik Hanafi's annual income in 2020 was RM70 000 and his *zakat* payment was RM500.

(a) Calculate Encik Hanafi's chargeable income.
(b) Calculate the income tax payable by Encik Hanafi.
(c) Discuss the difference between tax relief and tax rate.

Solution:

Understanding the problem

Annual income = RM70 000
Individual relief = RM 9 000
Life insurance and EPF relief = RM 6 000
Donation exemption = RM 500
Zakat payment = RM 500

Devising a strategy

(a) Determine the chargeable income, by taking into consideration tax exemption and tax relief.
(b) Calculate the income tax by referring to tax rate table and total rebate.
(c) Differentiate the meaning of tax relief and tax rebate.

Implementing the strategy

(a) Chargeable income = total annual income – tax exemption – tax relief
= RM70 000 – RM500 – (RM9 000 + RM6 000)
= RM54 500

(b) By referring to tax rate table:

Tax on the first RM50 000 = RM1 800
Tax on the next balance = (RM54 500 – RM50 000) × 14%
= RM4 500 × 14%
= RM630
Income tax payable = RM1 800 + RM630 – RM500
= RM1 930

(c) Tax relief is deducted from annual income. Tax rebate is deducted from income tax imposed.

Making a conclusion

(a) RM54 500
(b) RM1 930
(c) Tax relief is deducted from annual income, while tax rebate is deducted from income tax imposed.

Example 13

Mr Khor used 650 kWh of electricity in November. It is given that the usage exceeding 600 kWh within a month will be subjected to service tax of 6%. The table below shows Mr Khor's home electricity bill in November.

Tariff block (kWh)	Prorated block (kWh)	Rate (RM)	Amount (RM)
200	200	0.218	43.60
100	100	0.334	x
300	300	0.516	y
300	50	0.546	z

Info Bulletin

Example of electricity bill

bit.do/ElectricityBill

(a) Calculate the values of x, y and z.
(b) Calculate the total amount of payment which is not subjected to service tax.
(c) Calculate the service tax charged in the electricity bill in November.

Solution:

Understanding the problem

Total electricity usage
= 650 kWh

Service tax rate = 6%

Devising a strategy

(a) Calculate the values of x, y and z.
(b) Calculate the total amount of payment which is not subjected to service tax.
(c) Calculate the service tax charged.

Implementing the strategy

(a) $x = 100 \times 0.334$
$= 33.40$

$y = 300 \times 0.516$
$= 154.80$

$z = 50 \times 0.546$
$= 27.30$

(b) Total amount of payment which is not subjected to service tax
= RM43.60 + RM33.40 + RM154.80
= RM231.80

(c) Service tax charged = RM27.30 × 6%
= RM1.64

Making a conclusion

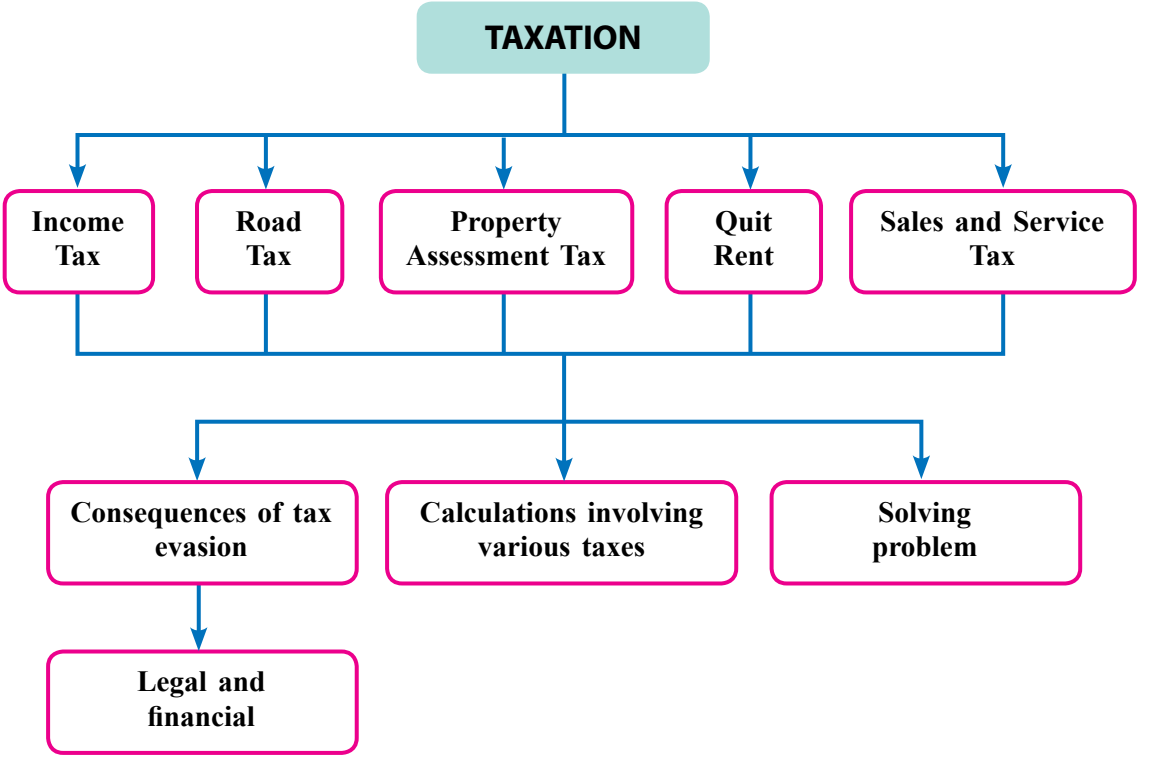
(a) $x = 33.40$, $y = 154.80$, $z = 27.30$
(b) RM231.80
(c) RM1.64

Self Practice 4.1d

1. Han and his wife received an annual salary of RM63 400 and RM52 100 respectively in 2020. It is given that the total tax relief that was eligible to be claimed by Han and his wife was RM61 219. Calculate the total income tax payable by Han and his wife by using joint tax assessment.

2. Encik Jamil is a manager in a company. His annual salary in 2020 was RM146 200. In that year, he paid *zakat* amounting to RM2 500. It is given that his total tax relief is RM46 250 and his *PCB* deduction is RM1 400 each month.
   (a) Calculate Encik Jamil's chargeable income.
   (b) Calculate the income tax payable by Encik Jamil.
   (c) Should Encik Jamil make any additional tax payment to IRB? Explain your answer.

Summary Arena



Reflection

At the end of this chapter, I can

		
explain the purpose of taxation.		
describe various taxes, hence the consequences of tax evasion from legal and financial aspects.		
investigate, interpret and perform calculations involving various taxes.		
solve problems involving taxation.		

MINI PROJECT

Taxpayers need to interpret, report and make income tax payment on annual basis. By using the latest tax exemption and tax relief items, you are required to create a folio about income tax assessment for one of your family members. Your folio should contain

(a) front page
(b) content: (i) Introduction of income tax
(ii) Latest tax exemption and tax relief items
(iii) Income tax assessment of a family member
(c) conclusion

Extensive Practice

Scan the QR code or visit bit.do/QuizE04 for interactive quiz

UNDERSTAND

1. Explain the purpose of tax as a tool in implementing government policy with an appropriate example.

2. State the entities responsible for collecting
(a) income tax
(b) property assessment tax
(c) road tax
(d) sales and service tax

3. _____ is a tax levied on income earned by an individual or a company in Malaysia.

4. State the consequences of evading property assessment tax.

5. Explain the meaning of tax relief for income tax.

6. State two types of tax rebate allowed by the government.

MASTERY

7. Azman owns a piece of land with an area of 6.7 m × 22.9 m, where his house is built. The state government has fixed the quit rent rate as RM0.40 per square metre. Calculate the quit rent payable by Azman each year.

8. Jordan's monthly salary was RM6 500. In year 2020, he received a bonus of one month salary. His tax exempt allowances amounting to RM9 600 were included in his salary. He also donated to a government-approved organisation amounting to RM1 000. It is given that the total tax relief was RM10 400. Calculate Jordan's chargeable income.

9. Encik Osman had a total chargeable income of RM35 800 in 2020. He has paid *zakat* amounting to RM700. Calculate the income tax payable by Encik Osman for that year.

10. Jun Kiat wants to buy a new car to be used in Peninsular Malaysia. He allocates RM280 for road tax payment for the new car. What is the maximum engine capacity of Jun Kiat's new car?

11. Mr Raju owns a service apartment in Ampang. The rental of the apartment is estimated at RM1 200 per month and he paid RM648 a year for property assessment tax. Calculate the property assessment tax rate for the service apartment.

12. Selvi bought a packet of *nasi lemak* from a stall for RM5.50, not inclusive of service tax. Then, she bought a cup of coffee from a coffee house for RM14.80, inclusive of 6% service tax.
    (a) Explain why sales and service tax was not collected in the stall.
    (b) Calculate the original price for a cup of coffee without sales and service tax.

13. Encik Suhaimi earned an annual salary of RM85 700 in 2020. His wife had no income. The following are the expenses to be claimed by Encik Suhaimi.

Item	Encik Suhaimi
Individual	RM9 000
Life insurance and EPF (limited to RM7 000)	RM7 760
Lifestyle (limited to RM2 500)	RM2 000
Spouse (tax relief for spouse who has no income)	RM4 000
Medical treatment expenses for parents (limited to RM5 000)	RM1 500

Calculate the total income tax for Encik Suhaimi and his wife by using joint tax assessment.

CHALLENGE

14. Eng Huat is a factory worker. His annual salary was RM45 000 a year. Similar to previous years, he donated RM500 to an approved medical healthcare centre. It is given that the total tax relief claimed was RM12 530.

(a) Calculate Eng Huat's chargeable income.

(b) Explain whether Eng Huat is eligible for tax rebate.

(c) Calculate the income tax payable by Eng Huat.

15. Puan Aida's annual income was RM84 400 in 2020. She claimed tax relief for individual for RM9 000, life insurance and EPF for RM7 000, lifestyle for RM1 200, medical insurance for RM2 800, medical treatment expenses for her parents for RM2 100 and an unmarried child aged under 18 years old for RM2 000. Puan Aida paid *zakat* of RM600 for that year.

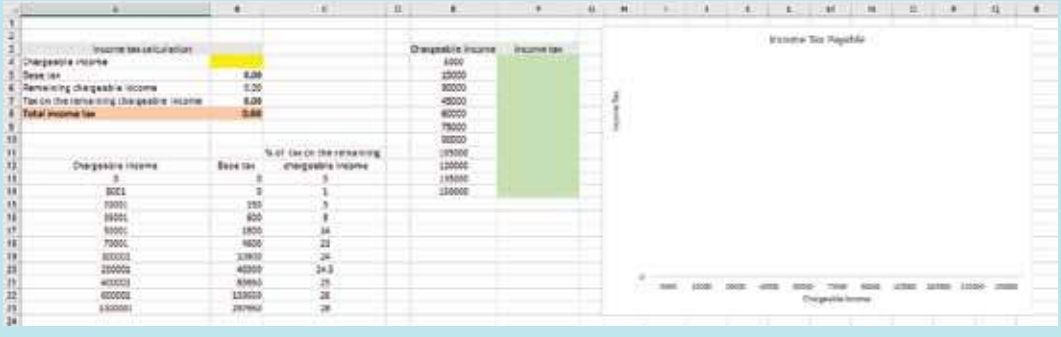
(a) Calculate the chargeable income for Puan Aida.

(b) Calculate the income tax payable by Puan Aida.

(c) If *PCB* of RM450 is deducted monthly from her salary, should Puan Aida make additional income tax payment?

EXPLORING MATHEMATICS

Electronic spreadsheet software can be used to simplify income tax calculation. By using the spreadsheet, calculate the income tax for chargeable income of RM5 000, RM15 000, RM30 000, …, RM150 000. Scan the QR code or visit bit.do/EMChap4 for your calculations.



Key in the value of chargeable income into the yellow box. Then, key in the corresponding income tax value into the green box.

Observe the Income Tax Payable graph on the right. Make a conclusion based on that graph.

CHAPTER 5

Congruency, Enlargement and Combined Transformations

What will you learn?

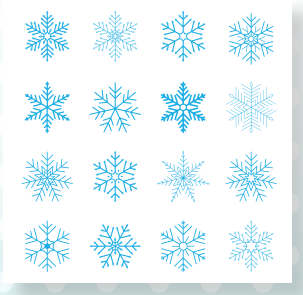
- Congruency
- Enlargement
- Combined Transformation
- Tessellation

Why study this chapter?

Astronomers use the telescope to observe the surface of a planet. The surface of the distant planet appears to be magnified through the telescope. However, the resulting image is inverted with its original object. Reflection is applied to get the actual picture.

Do you know?

Johannes Kepler (1571-1630) was a German mathematician and astronomer who had documented the study of tessellations in 1619. He used the concept of tessellation to explore and explain the structure of snowflakes.



For more information:

bit.do/DoYouKnowChap5

WORD BANK

scale factor	*faktor skala*
congruency	*kekongruenan*
similarity	*keserupaan*
reflection	*pantulan*
enlargement	*pembesaran*
rotation	*putaran*
tessellation	*teselasi*
transformation	*transformasi*
translation	*translasi*

Terengganu Drawbridge which is located in Kuala Terengganu is a new landmark for Terengganu in 2019. It is the longest drawbridge in the world with a length of 638 metres, connecting Kuala Terengganu and Kuala Nerus at the river mouth of Sungai Terengganu. The unique feature of the bridge is that it can be lifted in the centre to facilitate the passage of large ships beneath it. Observe the structure of the bridge. Which parts have congruent shapes?

5.1 Congruency

How to differentiate between congruent and non-congruent shapes?

Learning Standard

Differentiate between congruent and non-congruent shapes based on sides and angles.

Observe each of the diagrams below. What is your observation regarding the size and shape of each pair of objects shown?

Two pieces of tiles

Petronas Twin Towers

The front and rear tyres of a car

The pairs of objects in the diagrams above have the same size and shape regardless of their position and arrangement. The pairs of objects are **congruent**.

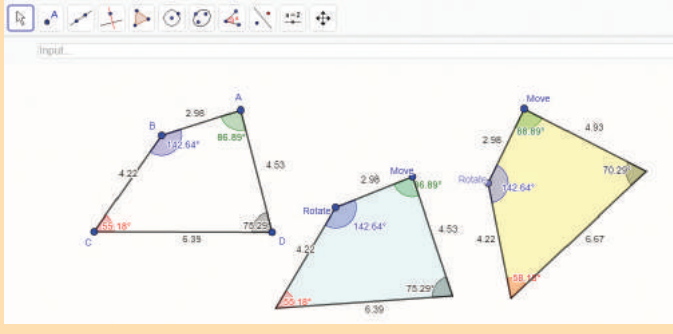
What are the characteristics used to identify that two objects are congruent?

MIND MOBILISATION 1 Pairs

Aim: To explore the congruent shapes based on sides and angles.

Steps:

1. Open the file GGB501 for this activity.



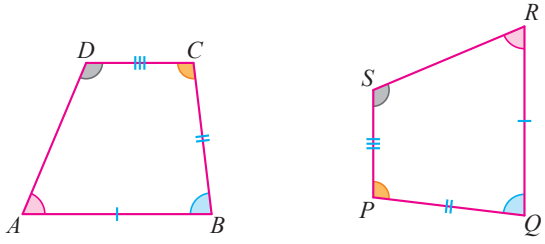
Scan the QR code or visit bit.do/GGB501BI to obtain the GeoGebra file for this activity.

2. Observe the measurements of the sides and angles for each of the quadrilaterals displayed.
3. Drag the points "Move" and "Rotate" to move the blue quadrilateral to overlap quadrilateral *ABCD*. Does the blue quadrilateral overlap quadrilateral *ABCD* completely? What is your observation on the measurements of the sides and angles?
4. Repeat step **3** for the yellow quadrilateral.

Discussion:

What is your conclusion about congruent shapes based on sides and angles?

The results of Mind Mobilisation 1 show that two **congruent** polygons have the same measurement for the corresponding sides and angles.



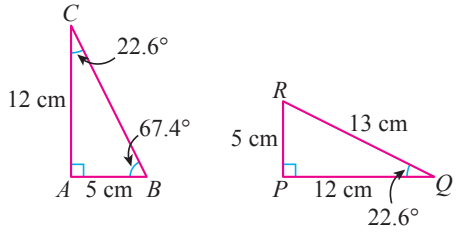
In the diagram above, quadrilaterals $ABCD$ and $PQRS$ are congruent with the conditions shown in the following table.

Lengths of corresponding sides	Sizes of corresponding angles
$AB = RQ$ $BC = QP$ $CD = PS$ $AD = RS$	$\angle BAD = \angle QRS$ $\angle ABC = \angle RQP$ $\angle BCD = \angle QPS$ $\angle ADC = \angle RSP$

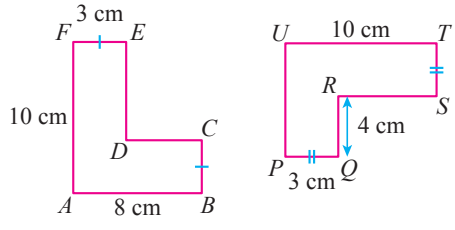
Example 1

Determine whether each pair of the following shapes are congruent.

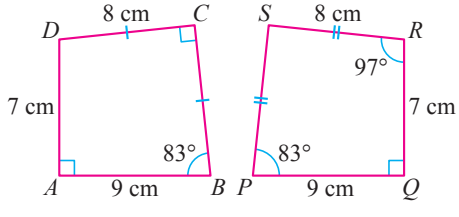
(a)



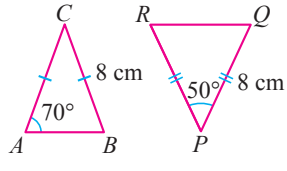
(b)



(c)



(d)



Solution:

(a) $BC = \sqrt{5^2 + 12^2}$
$= 13$ cm
$= RQ$

and

$\angle PRQ = 180° - 90° - 22.6°$
$= 67.4°$
$= \angle ABC$

The measurements of all corresponding sides and angles are equal.
Hence, both shapes are congruent.

(b) $DC = 8 - 3$ or $PU = 4 + 3$
$= 5$ cm $= 7$ cm
$\neq RQ$ $\neq BA$

Hence, both shapes are not congruent.

(c) $\angle ADC = 360° - 90° - 83° - 90°$
$= 97°$
$= \angle QRS$

The measurements of all corresponding sides and angles are equal. Hence, both shapes are congruent.

(d) $\angle ACB = 180° - 70° - 70°$
$= 40°$
$\neq \angle QPR$

Hence, both shapes are not congruent.

Smart TIPS

If any corresponding side or angle of the first shape is not equal to the second shape, both shapes are not congruent.

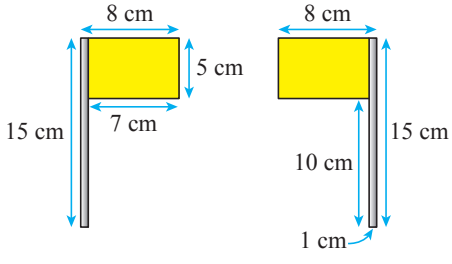
Info Bulletin

Triangles ABC and PQR which are congruent can be written as $\Delta ABC \cong \Delta PQR$.

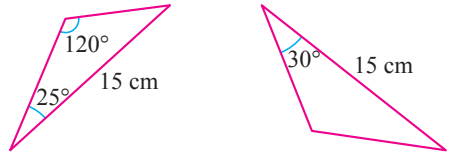
Self Practice 5.1a

1. Determine whether each pair of the following shapes are congruent.

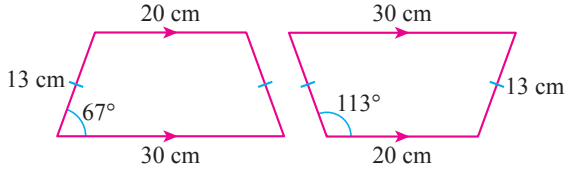
(a)



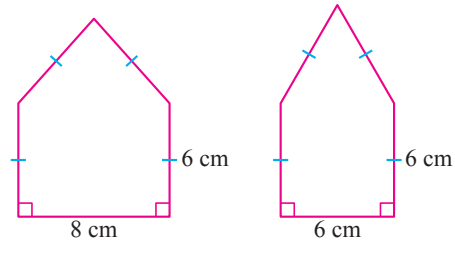
(b)



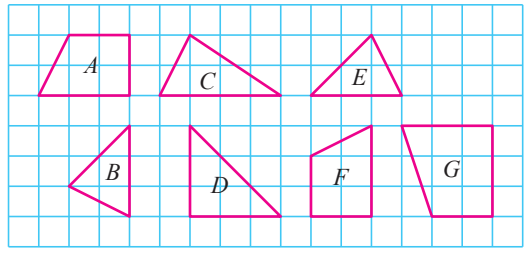
(c)



(d)



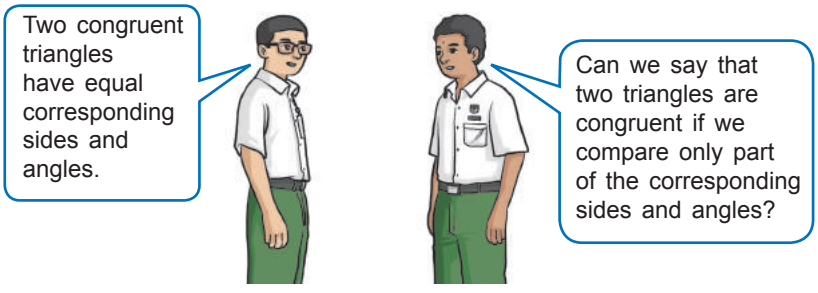
2. Determine pairs of congruent shapes in the diagram below.



What are the characteristics of triangle congruency?

Learning Standard

Make and verify the conjecture of triangle congruency based on sides and angles.

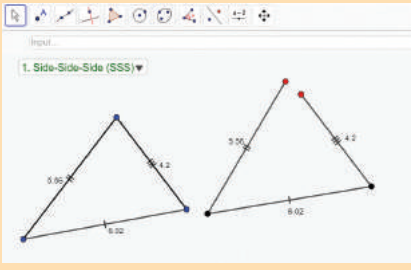


MIND MOBILISATION 2 Group

Aim: To make and verify the conjecture of triangle congruency.

Steps:

1. Open the file GGB502 for this activity.



Scan the QR code or visit bit.do/GGB502BI to obtain the GeoGebra file for this activity.

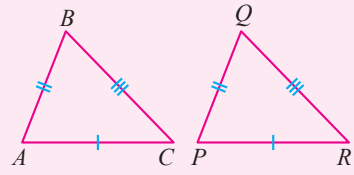
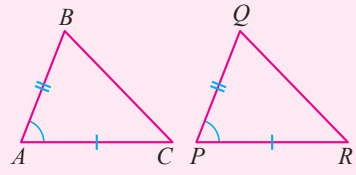
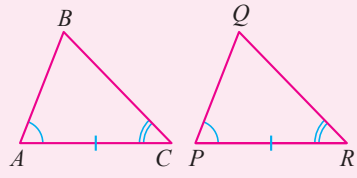
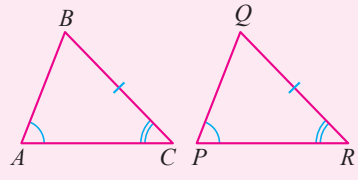
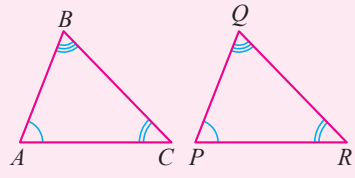
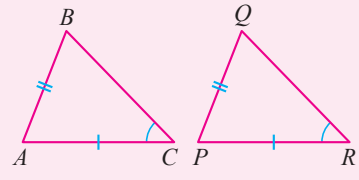
Scan the QR code or visit bit.do/WSChap5i to obtain the worksheet for this activity.

2. Select Side-Side-Side (SSS). Observe the measurements labelled on both diagrams.
3. Drag the red points to form a complete triangle. Drag the black points to change the position and orientation. Can you form congruent triangles? Can you find the non-congruent triangles?
4. Drag the blue points to change the shape of triangles and repeat step **3**.
5. Select Side-Angle-Side (SAS), Angle-Side-Angle (ASA), Angle-Angle-Side (AAS), Angle-Angle-Angle (AAA) and Side-Side-Angle (SSA) and repeat steps **2** to **4** for the exploration of other properties of congruent triangles.
6. Open the worksheet for this activity. Complete it based on your exploration.

	Can you form the congruent triangles?	Can you find the non-congruent triangles?	Conclusion
Side-Side-Side (SSS)			If _____, then the triangles are congruent.

Discussion:
What is your conclusion about triangle congruency based on the sides and angles?

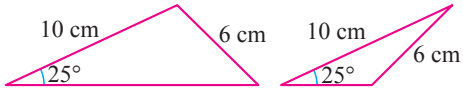
The results of Mind Mobilisation 2 show that congruent triangles have the following properties.

Properties of triangle congruency	
Side-Side-Side (SSS) • The length of corresponding sides are equal • $AC = PR$ $AB = PQ$ $BC = QR$	
Side-Angle-Side (SAS) • Two corresponding sides are equal in length and the size of the corresponding subtended angle between the two sides are equal • $AC = PR$ $\angle BAC = \angle QPR$ $AB = PQ$	
Angle-Side-Angle (ASA) • Two corresponding angles are equal and the length of the corresponding side between the two angles are equal • $\angle BAC = \angle QPR$ $AC = PR$ $\angle ACB = \angle PRQ$	
Angle-Angle-Side (AAS) • Two corresponding angles are equal and the length of one of the corresponding sides which does not lie between the two angles are equal • $\angle BAC = \angle QPR$ $\angle ACB = \angle PRQ$ $BC = QR$	
Angle-Angle-Angle (AAA) • All the three corresponding angles are equal • The area of the pair of triangles must be equal • $\angle BAC = \angle QPR$ $\angle ACB = \angle PRQ$ $\angle ABC = \angle PQR$	
Side-Side-Angle (SSA) • Two corresponding sides are equal in length and one of the corresponding angles which is not subtended between the two sides are equal • The area of the pair of triangles must be equal • $AC = PR$, $AB = PQ$, $\angle ACB = \angle PRQ$	

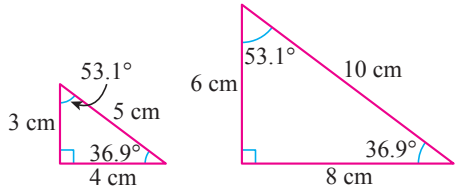
Example 2

Determine whether each pair of the following triangles satisfies the properties of triangle congruency. Justify your answer.

(a)



(b)



Solution:

(a) No. Although both triangles have two corresponding sides and one corresponding angle of the same measurement, the remaining side and angles are different in measurement. The areas are different.

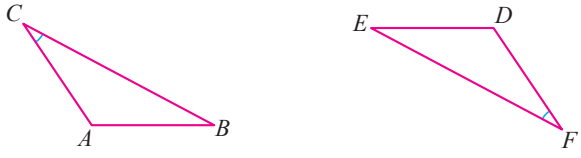
(b) No. Although both triangles have three corresponding angles of the same measurement, all the corresponding sides are different in measurement. The areas are different.

Smart TiPS

Side-Side-Angle (SSA) and Angle-Angle-Angle (AAA) cannot be used to determine two triangles are congruent except the areas of the triangles are the same.

Example 3

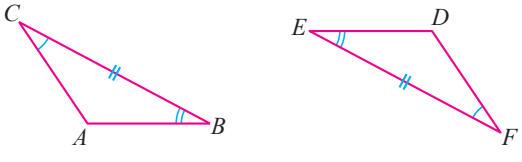
The diagram below shows two congruent triangles.



It is given that the triangle congruence rule used to determine both triangles are congruent is Angle-Side-Angle (ASA). Complete the table below with the other characteristics involving Angle-Side-Angle (ASA).

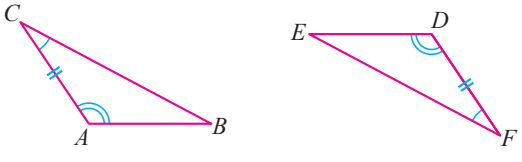
Angle	$\angle ACB = \angle DFE$
Side	
Angle	

Solution:



Angle	$\angle ACB = \angle DFE$
Side	$BC = EF$
Angle	$\angle ABC = \angle DEF$

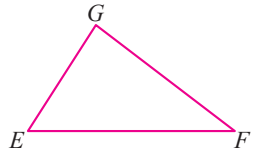
or



Angle	$\angle ACB = \angle DFE$
Side	$AC = DF$
Angle	$\angle BAC = \angle EDF$

Example 4

The diagram below shows a triangle EFG.



It is given that another triangle KLM has the same shape and size with triangle EFG. State the triangle congruence rule used to determine if both triangles are congruent.

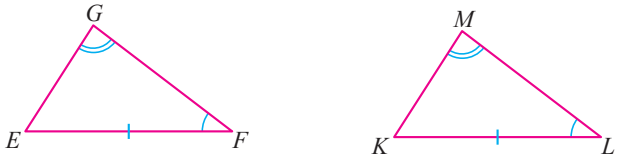
(a) $\angle EGF = \angle KML$, $\angle EFG = \angle KLM$, $EF = KL$

(b) $EG = KM$, $FG = LM$, $EF = KL$

(c) $EF = KL$, $\angle FEG = \angle LKM$, $EG = KM$

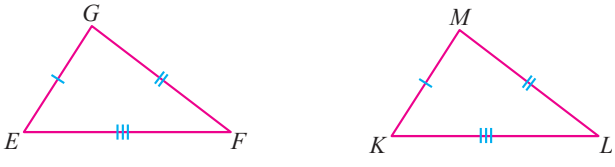
Solution:

(a)



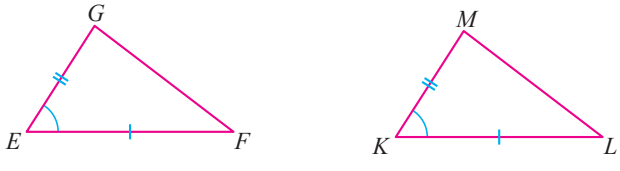
Two corresponding angles and one corresponding side that does not lie between the two angles are given. Therefore, the triangle congruence rule is Angle-Angle-Side (AAS).

(b)



Three corresponding sides are given. Therefore, the triangle congruence rule is Side-Side-Side (SSS).

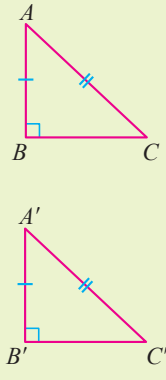
(c)



Two corresponding sides and one corresponding subtended angle between the two sides are given. Therefore, the triangle congruence rule is Side-Angle-Side (SAS).

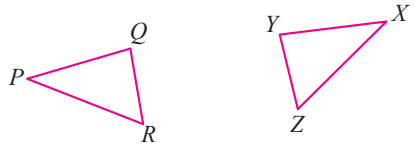
Info Bulletin

Right angle-Hypotenuse-Side (RHS) is also a special congruence property for right-angled triangle. If the hypotenuse and one of the corresponding non-hypotenuse sides of two right-angled triangles are equal in length, then the two triangles are congruent. This congruence property is based on Side-Side-Angle (SSA) rule.



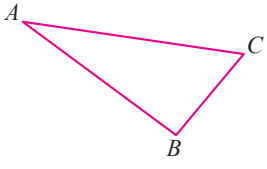
Self Practice 5.1b

1. The diagram below shows two triangles, PQR and XYZ.



It is given that both triangles PQR and XYZ are congruent. If $PQ = XY$, state the other characteristics involved if each of the following is used
(a) Side-Angle-Side
(b) Side-Side-Angle

2. The diagram on the right shows a triangle ABC. It is given that another triangle PQR is congruent with the triangle ABC. State the triangle congruence rule involved if
(a) $AB = PQ$, $BC = QR$ and $AC = PR$
(b) $AB = PQ$, $\angle ABC = \angle PQR$ and $\angle BAC = \angle QPR$



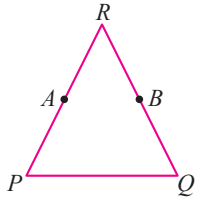
How to solve problems involving congruency?

Learning Standard

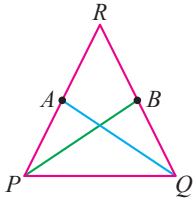
Solve problems involving congruency.

Example 5

The diagram on the right shows an isosceles triangle PQR where $PR = QR$. A and B are the midpoints of the sides PR and QR respectively. Show that triangles PBR and QAR are congruent.



Solution:



Based on the diagram on the left,
- $PR = QR$
- $\angle PRB = \angle QRA$
- $BR = AR$

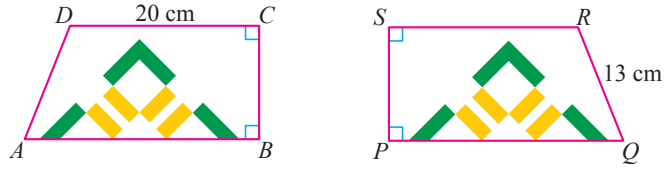
Triangles PBR and QAR satisfy the properties of Side-Angle-Side (SAS). Therefore, triangles PBR and QAR are congruent.

Application & Career

In the field of engineering, the concept of triangle congruence is used to build the bridge supports so that the bridge can be supported in a balanced state.

Example 6

Siva prepared two pieces of congruent trapezium shaped cards as shown in the diagram below. Each card is painted with half of a logo.



Siva put the two pieces of card together to form a pentagon with a complete logo. If the perimeter of the pentagon is 90 cm, calculate the area, in cm^2, of the pentagon.

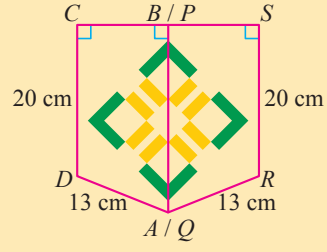
Solution:

Method 1

Understanding the problem

- Trapezium $ABCD$ and trapezium $QPSR$ are congruent.
- $AB = QP$, $BC = PS$, $CD = SR$, $AD = QR$.
- The perimeter of the pentagon formed is 90 cm.
- Calculate the area of the pentagon formed.

Devising a strategy



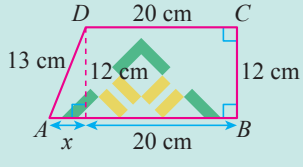
- $CS = 90 \text{ cm} - AD - DC - QR - RS$
- Calculate the length of AB
- Area of pentagon
  $= 2 \times$ area of trapezium

Implementing the strategy

$CS = 90 - 13 - 20 - 13 - 20$
$= 24$ cm

$CB = 12$ cm

$x = \sqrt{13^2 - 12^2}$
$= 5$ cm



$AB = 20 + 5$
$= 25$ cm

Area of pentagon $= 2 \times \frac{1}{2}(20 + 25)(12)$
$= 540$ cm^2

Making a conclusion

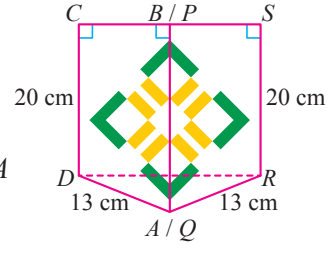
Hence, the area of the pentagon is 540 cm^2.

Method 2

Area of rectangle $CDRS = 24 \times 20$
$= 480$ cm^2

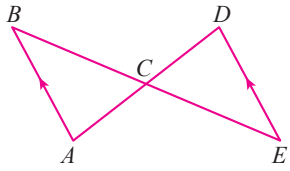
Area of triangle $DRA = \frac{1}{2}(24)(5)$
$= 60$ cm^2

Area of pentagon = area of rectangle $CDRS$ + area of triangle DRA
$= 480 + 60$
$= 540$ cm^2

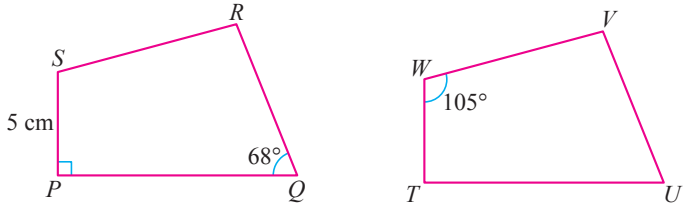


Self Practice 5.1c

1. In the diagram on the right, the lines AB and DE are parallel. BE and AD are straight lines. C is the midpoint of AD. Show that triangles ABC and DEC are congruent.

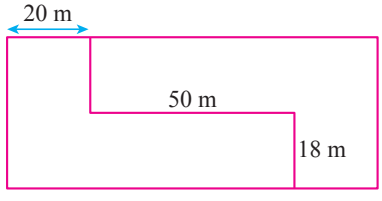


2. The diagram below shows two congruent quadrilaterals.



Given that the length of diagonal WU is 13 cm, calculate
(a) the length, in cm, of PQ,
(b) $\angle SRQ$.

3. Muizuddin divided his rectangular piece of land into two congruent sections as shown in the diagram on the right. Calculate
(a) the perimeter, in m, of each section of the land,
(b) the area, in m^2, of each section of the land.



5.2 Enlargement

What is the meaning of similarity of geometric objects?

Learning Standard

Explain the meaning of similarity of geometric objects.

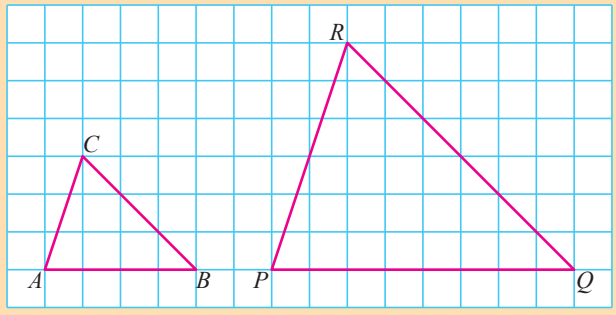
The photo below shows the house models which are **similar** to the real houses. This means that the house models have the **same shape** as the real houses despite the **different size**. What are the properties of two similar geometric objects?

MIND MOBILISATION 3 Pairs

Aim: To recognise the similarity of geometric objects.
Materials: Ruler and protractor

Steps:

1. By using ruler and protractor, measure the angles on each vertex and the length of each side of the two triangles in the following diagram.



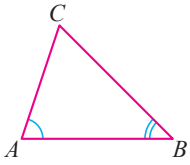
2. Complete the table below with your measurements.

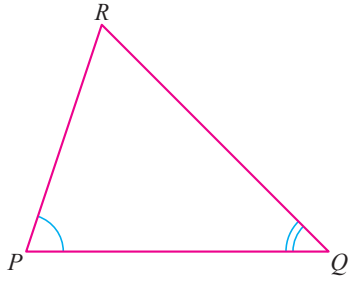
Size of angle		Length of side		Ratio of the corresponding sides
Triangle ABC	Triangle PQR	Triangle ABC	Triangle PQR	
$\angle A =$	$\angle P =$	$AB =$	$PQ =$	$\frac{PQ}{AB} =$
$\angle B =$	$\angle Q =$	$BC =$	$QR =$	$\frac{QR}{BC} =$
$\angle C =$	$\angle R =$	$CA =$	$RP =$	$\frac{RP}{CA} =$

Discussion:
Both of the triangles are similar. State your conclusion about
(a) the corresponding angles of both triangles,
(b) the corresponding sides of both triangles.

The results of Mind Mobilisation 3 show that two geometric objects are **similar** when

- all the corresponding angles are equal, namely $\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$
- all the ratios of corresponding sides are equal, namely $\frac{PQ}{AB} = \frac{QR}{BC} = \frac{RP}{CA}$





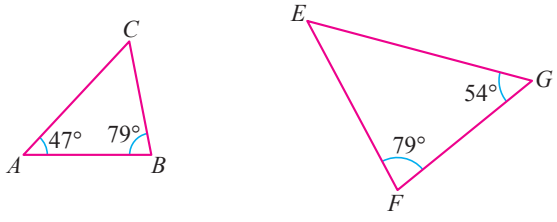
Smart TiPS

If all the corresponding sides of a pair of triangles are proportional, then all the corresponding angles are equal and vice versa.

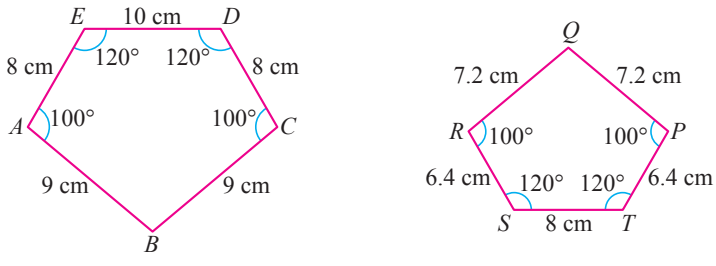
Example 7

Determine whether each pair of the following geometric objects are similar.

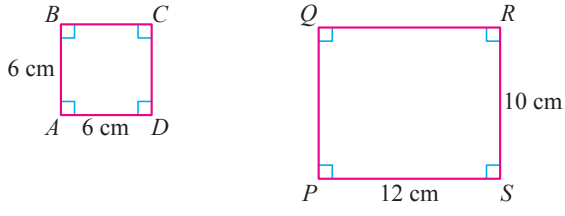
(a)



(b)



(c)



Solution:

(a) $\angle C = 180° - 47° - 79°$
$= 54°$
$= \angle G$
$\angle B = \angle F = 79°$
$\angle A = \angle E = 47°$

For a pair of triangles, all the corresponding sides are proportional when all the corresponding angles are equal. Hence, triangle ABC and triangle EFG are similar.

(b) All the corresponding angles are equal.

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{9}{7.2} = \frac{5}{4}$$
$$\frac{AE}{PT} = \frac{CD}{RS} = \frac{8}{6.4} = \frac{5}{4}$$
$$\frac{ED}{TS} = \frac{10}{8} = \frac{5}{4}$$

All the ratios of the corresponding sides are equal. Hence, pentagon $ABCDE$ and pentagon $PQRST$ are similar.

(c) All the corresponding angles are equal.

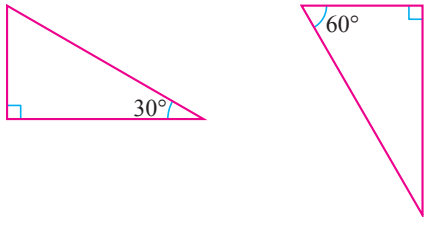
$$\frac{AB}{PQ} = \frac{DC}{SR} = \frac{6}{10} = \frac{3}{5}$$
$$\frac{AD}{PS} = \frac{BC}{QR} = \frac{6}{12} = \frac{1}{2}$$

The ratios of corresponding sides are not equal. Hence, quadrilateral $ABCD$ and quadrilateral $PQRS$ are not similar.

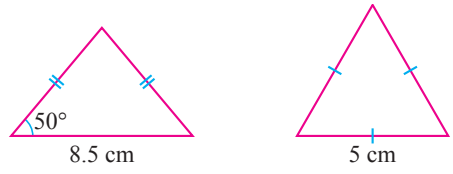
Self Practice 5.2a

1. Determine whether each pair of the following geometric objects are similar.

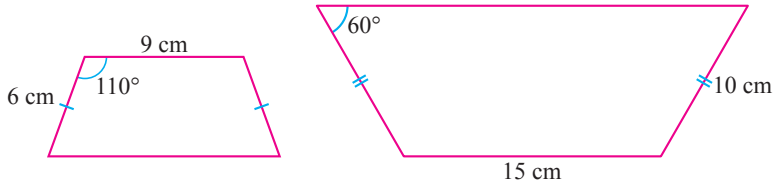
(a)



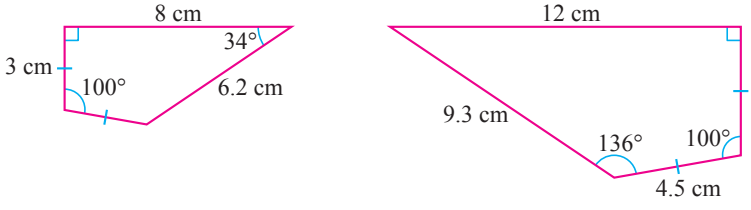
(b)



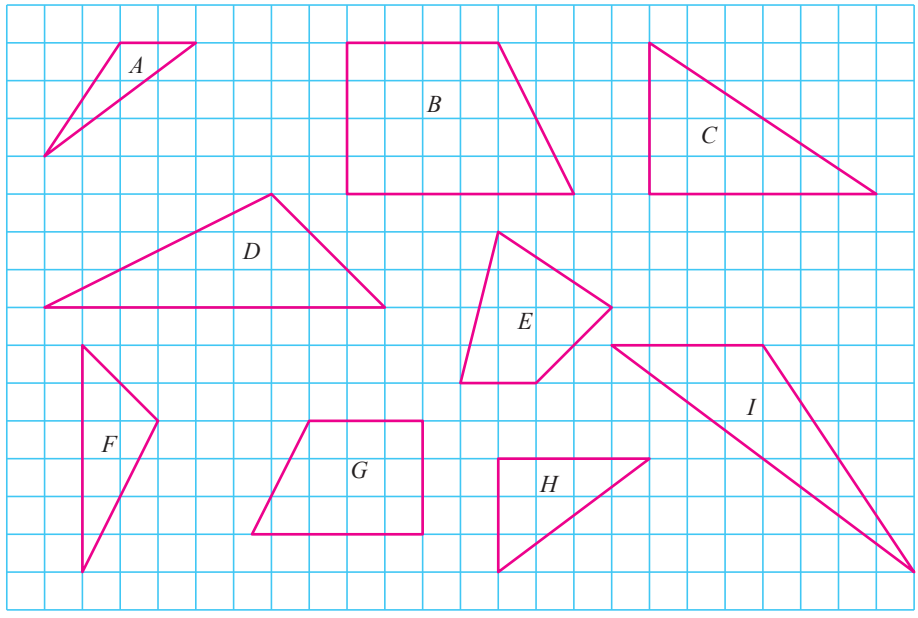
(c)



(d)



2. Determine pairs of similar objects in the diagram below.



What is the relationship between similarity and enlargement and how to describe enlargement?

Learning Standard

Make a connection between similarity and enlargement, hence describe enlargement using various representations.

A biologist uses a microscope to observe plant cells. The image produced through a microscope is thousands of times larger than its object. Do the object and the image produced satisfy the property of similarity?

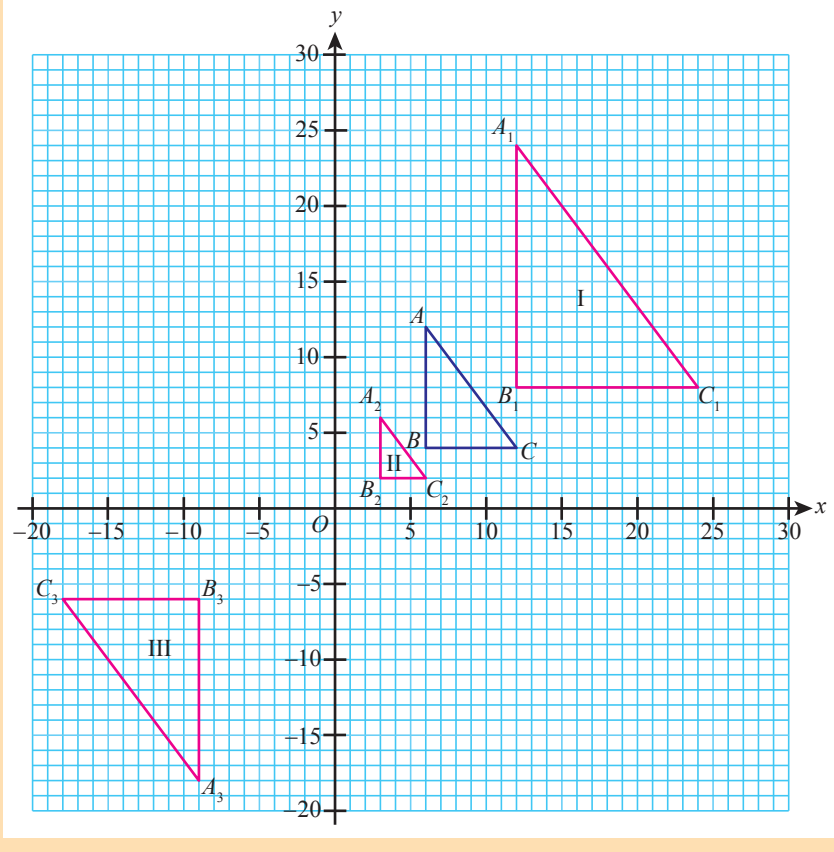
MIND MOBILISATION 4 Pairs

Aim: To make the connection between similarity and enlargement.

Materials: Ruler and protractor

Steps:

1. Observe the diagram below where $\Delta A_1B_1C_1$, $\Delta A_2B_2C_2$ and $\Delta A_3B_3C_3$ are the images of ΔABC under an enlargement.



2. Connect the corresponding points of all the images. Mark the intersection point between the three straight lines as point P.

3. Measure the sides, in units, and angles for each triangle and complete the following table.

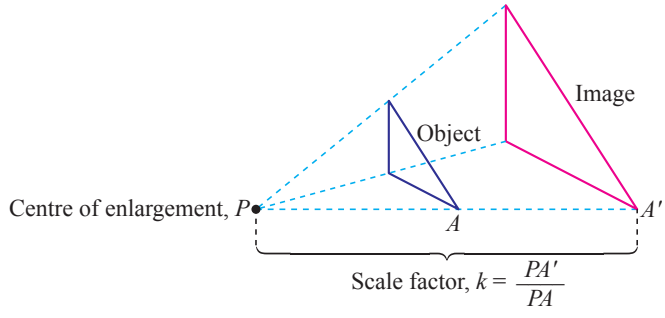
Triangle	Ratio of the corresponding sides	Ratio of the distance of vertex from point P	Equal corresponding angles? (Yes / No)	Similar with the triangle ABC? (Yes / No)
I	$\frac{A_1B_1}{AB} =$	$\frac{A_1P}{AP} =$		
	$\frac{B_1C_1}{BC} =$	$\frac{B_1P}{BP} =$		
	$\frac{A_1C_1}{AC} =$	$\frac{C_1P}{CP} =$		
II	$\frac{A_2B_2}{AB} =$	$\frac{A_2P}{AP} =$		
	$\frac{B_2C_2}{BC} =$	$\frac{B_2P}{BP} =$		
	$\frac{A_2C_2}{AC} =$	$\frac{C_2P}{CP} =$		
III	$\frac{A_3B_3}{AB} =$	$\frac{A_3P}{AP} =$		
	$\frac{B_3C_3}{BC} =$	$\frac{B_3P}{BP} =$		
	$\frac{A_3C_3}{AC} =$	$\frac{C_3P}{CP} =$		

Discussion:

1. What can you say about the ratio of the corresponding sides and the corresponding angles between the triangle ABC and its image?
2. Determine whether each image is similar to the triangle ABC.
3. Make a conclusion based on your findings.

The results of Mind Mobilisation 4 show that the object and image under an enlargement are similar.

Enlargement is a transformation in which all the points of an object move from a fixed point with a constant ratio. The fixed point is known as the **centre of enlargement** and the constant ratio is known as the **scale factor**.



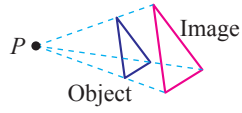
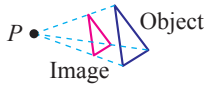
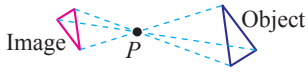
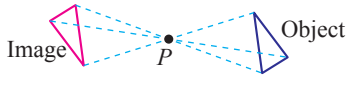
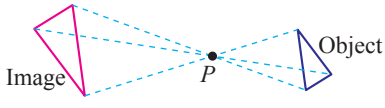
i-Technology

Scan the QR code or visit bit.do/GGB503BI to explore scale factor of enlargement.

In general, the scale factor k, of an enlargement can be determined as follows.

$$k = \frac{\text{distance of point of image from } P}{\text{distance of point of object from } P} \quad \text{or} \quad k = \frac{\text{length of side of image}}{\text{length of side of object}}$$

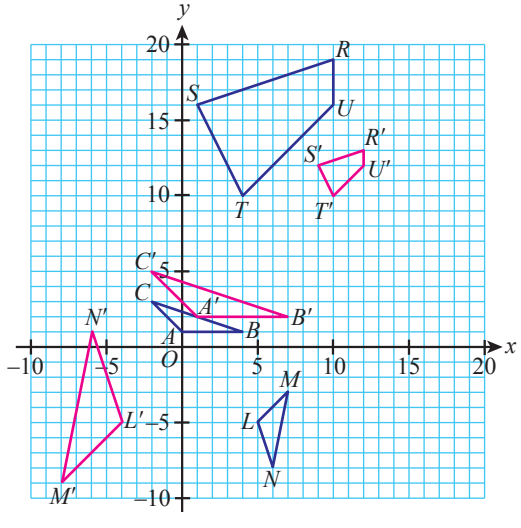
Different scale factors have different effects on enlargements:

Scale factor, k	Size of image	Position of image to the centre of enlargement P	
$k > 1$	**Larger** than the object	Be on the same side as the object	
$k = 1$	**Equal** in size to the object	Be on the same side as the object	
$0 < k < 1$	**Smaller** than the object	Be on the same side as the object	
$-1 < k < 0$	**Smaller** than the object	Be on the opposite side of the object	
$k = -1$	**Equal** in size to the object	Be on the opposite side of the object	
$k < -1$	**Larger** than the object	Be on the opposite side of the object	

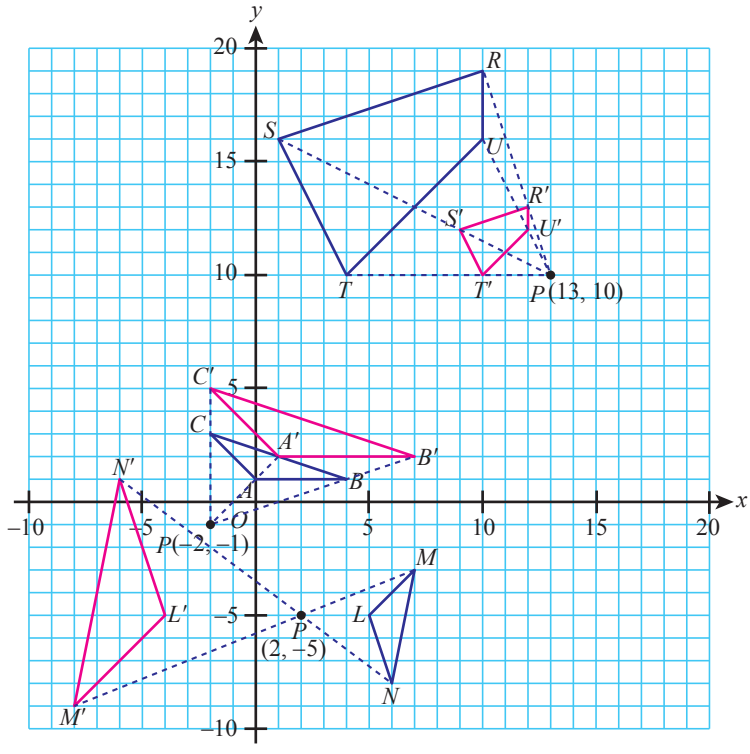
Example 8

The diagram on the right shows three objects and their images under the transformation of enlargement. Describe the enlargement by determining the scale factor and the centre of enlargement for the following:

(a) object ABC
(b) object $RSTU$
(c) object LMN



Solution:



(a) Scale factor $= \frac{A'B'}{AB}$

$= \frac{6}{4}$

$= \frac{3}{2}$ ← $k > 1$, image is larger than object

$A'B'C'$ is the image of ABC under an enlargement at centre $(-2, -1)$ with a scale factor of $\frac{3}{2}$.

(b) Scale factor $= \frac{U'R'}{UR}$

$= \frac{1}{3}$ ← $0 < k < 1$, image is smaller than object

$R'S'T'U'$ is the image of $RSTU$ under an enlargement at centre $(13, 10)$ with a scale factor of $\frac{1}{3}$.

(c) Scale factor $= \frac{\text{distance of } L' \text{ from centre of enlargement}}{\text{distance of } L \text{ from centre of enlargement}}$

$= -\frac{6}{3}$ ← The negative sign shows that the image is at the opposite side of the object. $k < -1$, image is larger than object.

$= -2$

$L'M'N'$ is the image of LMN under an enlargement at centre $(2, -5)$ with a scale factor of -2.

Smart TiPS

The centre of enlargement can be determined from the intersection point between all the straight lines which are connecting each pair of the corresponding points.

Info Bulletin

In enlargement, each pair of the corresponding sides of the object and the image are parallel.

Smart TiPS

When describing an enlargement, we need to state the **centre of enlargement** and the **scale factor**.

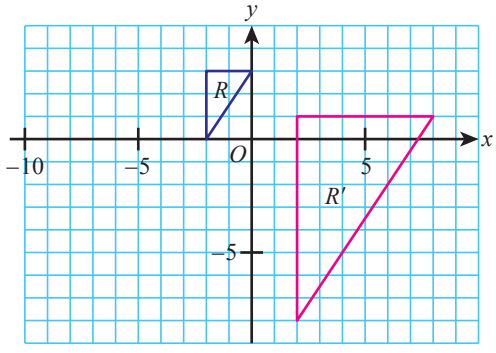
Smart TiPS

Negative scale factor will cause the image to appear on the other side of the centre of enlargement opposite to the object and the image will be inverted.

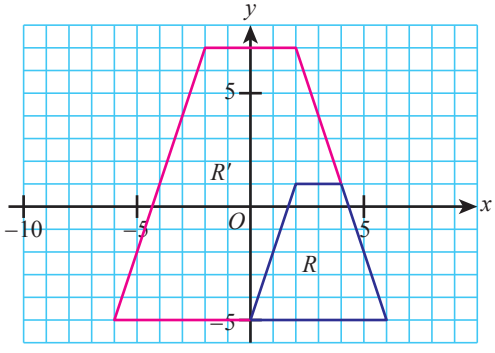
Self Practice 5.2b

1. It is given that R' is the image of object R. Describe the enlargement for each of the following.

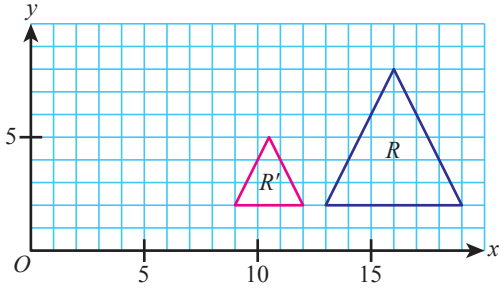
(a)



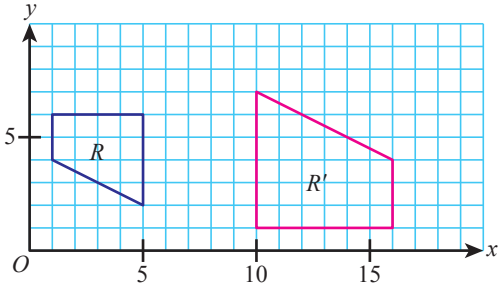
(b)



(c)

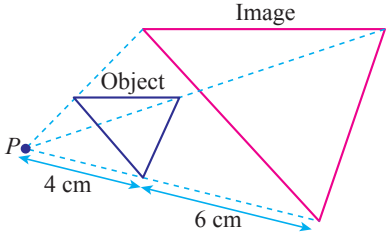


(d)

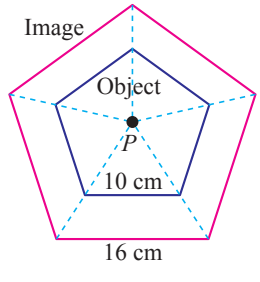


2. Describe the enlargement in each of the following diagrams.

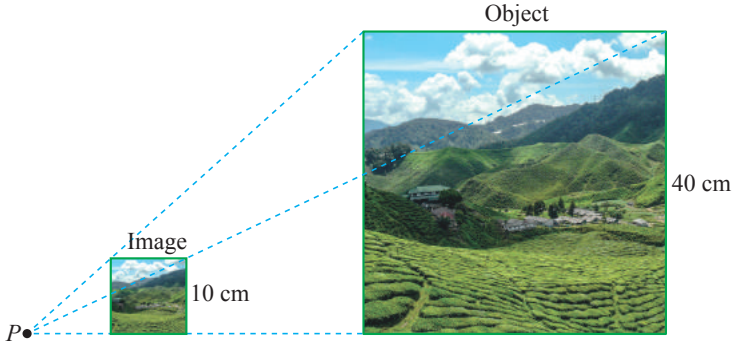
(a)



(b)



(c)

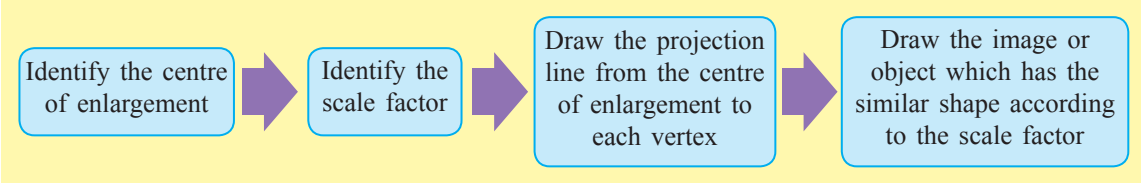


How to determine the image and object of an enlargement?

Learning Standard

Determine the image and object of an enlargement.

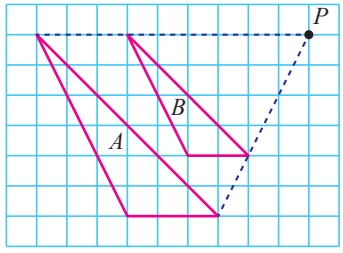
The flow chart below shows the steps to determine the image or object of an enlargement.



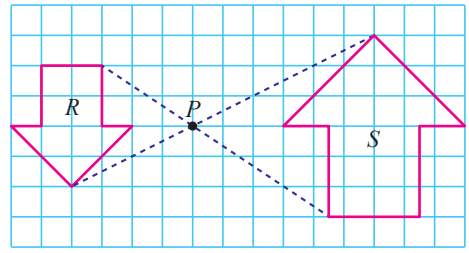
Example 9

Based on the given scale factor, determine the object and image for each of the following enlargements.

(a) Scale factor, $k = \frac{2}{3}$



(b) Scale factor, $k = -1.5$



Solution:

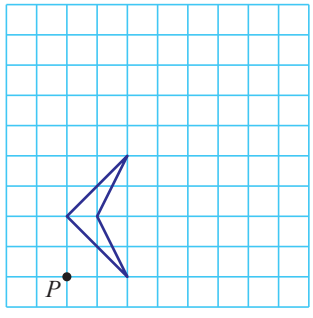
(a) When $k = \frac{2}{3}$, size of image is smaller than the object. A is object and B is image.

(b) When $k = -1.5$, size of image is larger than the object. R is object and S is image.

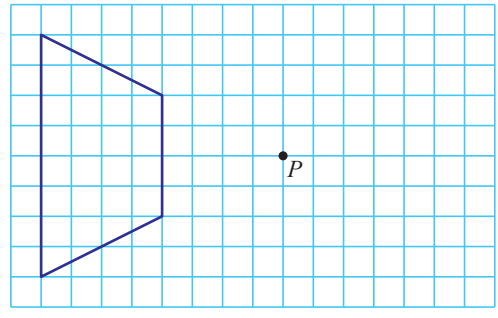
Example 10

Draw the image for each of the following objects using P as the centre of enlargement for the following scale factors.

(a) $k = 2$

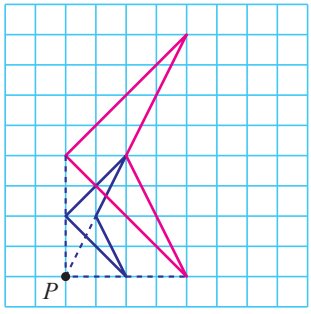


(b) $k = -\frac{3}{4}$

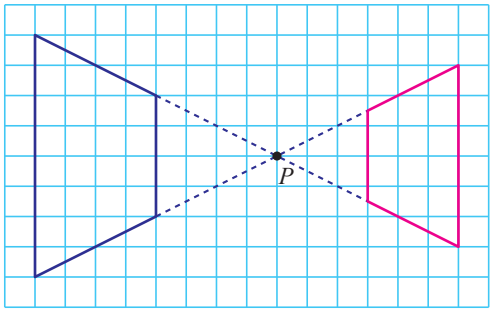


Solution:

(a) When $k = 2$, the distance of each vertex of the image from P is 2 times the distance of corresponding vertex of the object from P in the same direction.

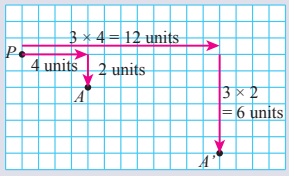


(b) When $k = -\frac{3}{4}$, the distance of each vertex of the image from P is $\frac{3}{4}$ times the distance of corresponding vertex of the object from P in the opposite direction.



Smart TiPS

We can use the horizontal distance and vertical distance to determine the ratio of the distance of a point from the centre of enlargement as the example below.



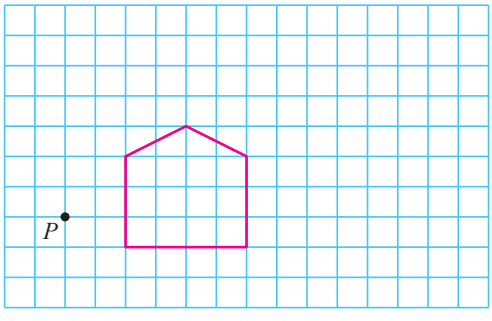
It is given that point A is object and point A' is its image, P is centre of enlargement and $k = 3$.

When $k = 3$, the vertical distance and horizontal distance of A' from P to the vertical distance and horizontal distance of A from P follow the ratio 3 : 1.

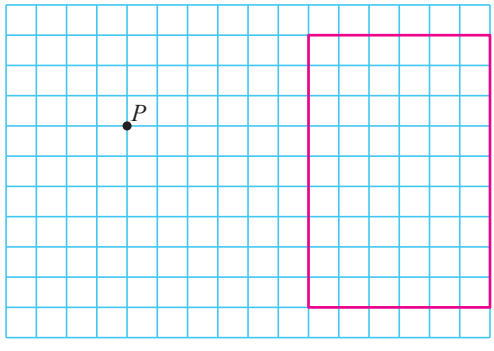
Example 11

Draw the object for each of the following images using P as the centre of enlargement for the following scale factors.

(a) $k = \frac{1}{2}$

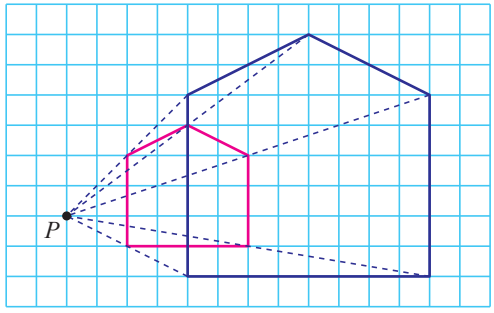


(b) $k = -3$

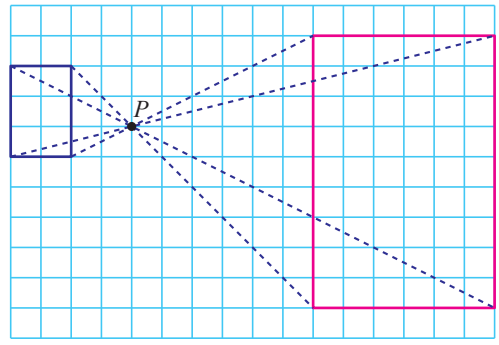


Solution:

(a) When $k = \frac{1}{2}$, the distance of each vertex of the object from P is 2 times the distance of corresponding vertex of the image from P in the same direction.



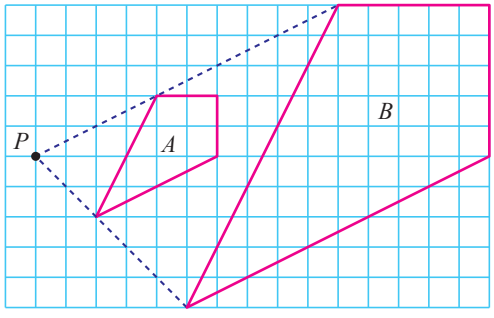
(b) When $k = -3$, the distance of each vertex of the object from P is $\frac{1}{3}$ times the distance of corresponding vertex of the image from P in the opposite direction.



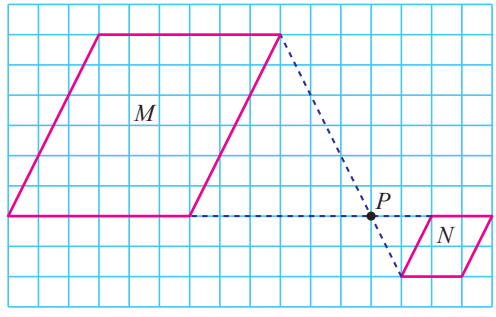
Self Practice 5.2c

1. Based on the given scale factor, determine the object and image for the following enlargements.

(a) Scale factor, $k = \frac{5}{2}$

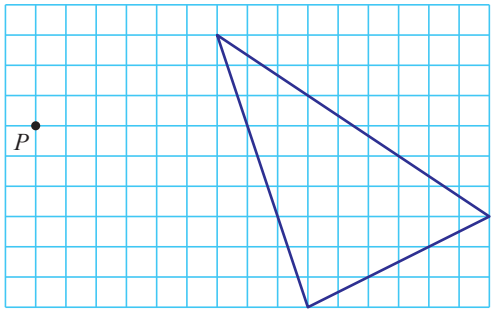


(b) Scale factor, $k = -3$

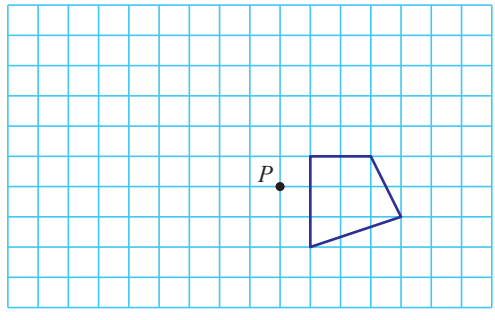


2. Draw the image for each of the following objects under the enlargement at centre P based on the scale factor given.

(a) $k = \frac{1}{3}$

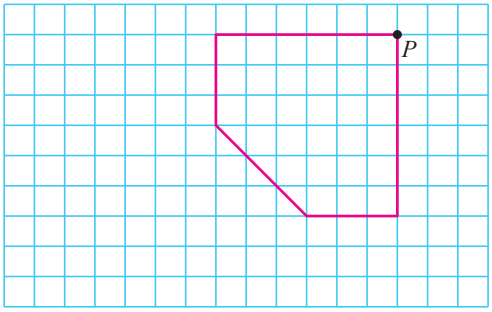


(b) $k = -2$

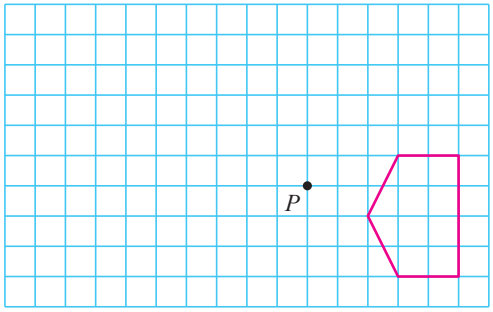


3. Draw the object for each of the following images under the enlargement at centre P based on the scale factor given.

(a) $k = \frac{3}{4}$



(b) $k = -\frac{1}{2}$



What is the relation between area of the image and area of the object of an enlargement?

Learning Standard

Make and verify conjecture on the relation between area of the image and area of the object of an enlargement.

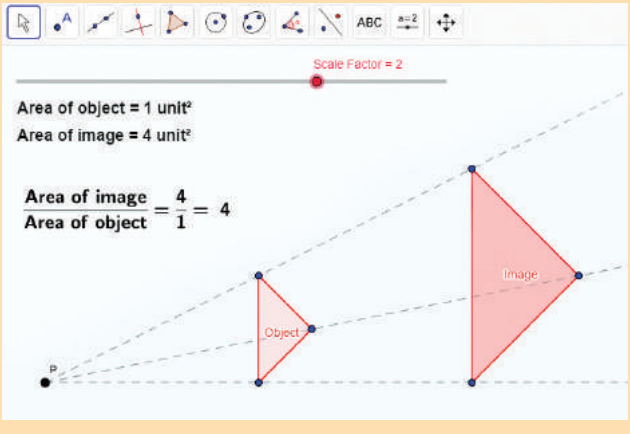
During enlargement, the lengths of corresponding sides are constantly proportional. What is the relation between area of the image and area of the object of an enlargement?

MIND MOBILISATION 5 Pairs

Aim: To explore the relation between area of the image and area of the object of an enlargement.

Steps:

1. Open the file GGB504 for this activity.



Scan the QR code or visit bit.do/GGB504BI to obtain the GeoGebra file for this activity.

2. Drag the slider 'Scale Factor' to some different values to observe the area of the object, area of the image and ratio of area of the image to area of the object.

3. Hence, complete the following table.

Scale factor, k	Area of the object (unit^2)	Area of the image (unit^2)	Ratio of area of the image to area of the object, $\frac{\text{area of the image}}{\text{area of the object}}$	k^2
$k = 2$	1	4	$\frac{4}{1} = 4$	$(2)^2 = 4$
$k = 3$				
$k = 4$				
$k = -1$				
$k = -2$				

Discussion:

1. What is the connection between scale factor and the ratio of area of the image to area of the object?
2. What is the relation between area of the image and area of the object of an enlargement?

The results of Mind Mobilisation 5 show that $\frac{\text{area of the image}}{\text{area of the object}} = k^2$ where k is the scale factor.

Hence, we can determine the area of the image of an enlargement with the formula:

Area of the image = k^2 × Area of the object

Example 12

The table below shows the different values of area of the object, area of the image and scale factor under enlargement. Complete the table.

	Area of the object	Area of the image	Scale factor, k
(a)	5 cm^2	45 cm^2	
(b)	12 unit^2		$\frac{7}{2}$
(c)		100 m^2	-4

Solution:

(a) $k^2 = \frac{\text{area of the image}}{\text{area of the object}}$

$= \frac{45}{5}$

$= 9$

$k = \sqrt{9}$

$= +3$ or -3

(b) $\left(\frac{7}{2}\right)^2 = \frac{\text{area of the image}}{12}$

area of the image $= \frac{49}{4} \times 12$

$= 147$ unit^2

(c) $(-4)^2 = \frac{100}{\text{area of the object}}$

area of the object $= \frac{100}{16}$

$= 6.25$ m^2

Self Practice 5.2d

1. The table below shows the different values of area of the object, area of the image and scale factor under enlargement. Complete the table.

Area of the object	Area of the image	Scale factor, k
18 unit^2	72 unit^2	
54 m^2		$\frac{1}{3}$
	31.25 cm^2	$\frac{5}{4}$

How to solve problems involving enlargement?

Learning Standard

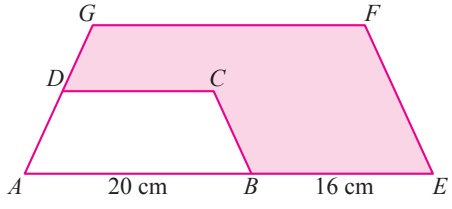
Solve problems involving enlargement.

Example 13

In the diagram on the right, trapezium $AEFG$ is the image of trapezium $ABCD$ under an enlargement at centre A. Given that the area of the coloured region is 168 cm^2, calculate

(a) the scale factor of the enlargement,

(b) the area, in cm^2, of trapezium $AEFG$.



Solution:

(a) Scale factor, $k = \frac{AE}{AB}$

$= \frac{20 + 16}{20}$

$= \frac{9}{5}$

(b) Let the area of trapezium $ABCD = L_1$ and the area of trapezium $AEFG = L_2$

$L_2 - L_1 = 168$

$L_2 - \frac{L_2}{\left(\frac{9}{5}\right)^2} = 168$

Area of image $= k^2 \times$ Area of object

$L_2 = \left(\frac{9}{5}\right)^2 \times L_1$

$L_1 = \frac{L_2}{\left(\frac{9}{5}\right)^2}$

$L_2 - \frac{25}{81}L_2 = 168$

$\frac{56}{81}L_2 = 168$

$L_2 = 243$

Hence, the area of trapezium $AEFG = 243$ cm^2.

Alternative Method

$L_2 = x$

$L_1 = \frac{x}{\left(\frac{9}{5}\right)^2}$

$= \frac{25}{81}x$

$L_2 - L_1 = 168$

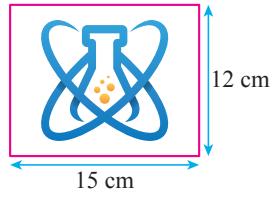
$x - \frac{25}{81}x = 168$

$\frac{56}{81}x = 168$

$x = 243$ cm^2

Example 14

The diagram below shows a rectangular logo drawn by a member of STEM Association.



On the school entrepreneur day, the STEM Association intends to put up a banner at their stall. The banner is designed based on the enlargement of the logo. If the area of the banner is 0.8 m^2, what are the measurements of the length and width, in cm, of the banner?

Solution:

Area of the logo = $15 \times 12 = 180$ cm^2 and area of the banner = 0.8 m^2 = $8\ 000$ cm^2.

$$k^2 = \frac{\text{area of the image}}{\text{area of the object}}$$

$$= \frac{8\ 000\ cm^2}{180\ cm^2}$$

$$= \frac{400}{9}$$

$$k = \frac{20}{3}$$

$$k = \frac{\text{length of side of image}}{\text{length of side of object}}$$

$$\frac{20}{3} = \frac{\text{Length}}{15}$$

$$\text{Length} = 15 \times \frac{20}{3}$$

$$= 100 \text{ cm}$$

$$\text{Width} = \frac{\text{Area}}{\text{Length}}$$

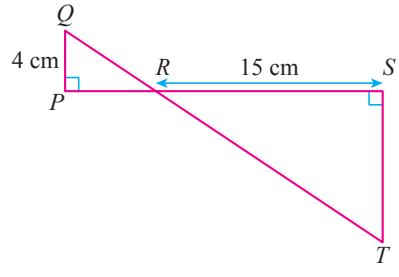
$$= \frac{8\ 000}{100}$$

$$= 80 \text{ cm}$$

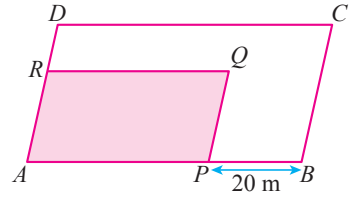
The measurement of the banner is 100 cm in length and 80 cm in width.

Self Practice 5.2e

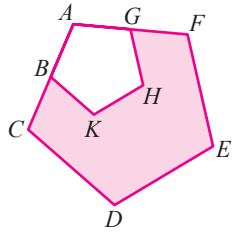
1. The diagram on the right shows two right-angled triangles where ΔPQR is the image of ΔSTR under an enlargement. It is given that $5PR = 2RS$. Calculate
(a) the scale factor of the enlargement,
(b) the area, in cm^2, of the whole diagram.



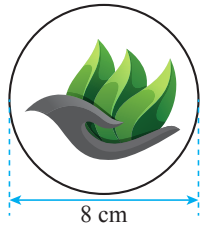
2. The diagram on the right shows the plan of a garden in the shape of a parallelogram. It is given that $ABCD$ is the image of $APQR$ with the enlargement at centre A and the area of the garden is 1 350 m^2. If the area of the coloured region is 600 m^2, calculate the length, in m, of AP.



3. In the diagram on the right, the pentagon $ACDEF$ is the image of pentagon $ABKHG$ under an enlargement where the vertex A is the centre of enlargement. It is given that B is the midpoint of side AC and the area of the pentagon $ABKHG$ is 17 cm^2. Calculate the area, in cm^2, of the coloured region.



4. The diagram on the right shows a circular badge used in an environmental protection programme which is launched by the Environmental Association. This association intends to paint a mural on the wall of a building based on the enlargement of the badge. It is given that the area of the mural is 4π m^2. What is the scale factor of the enlargement?



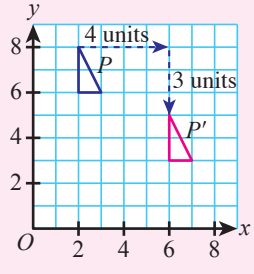
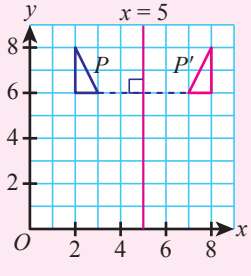
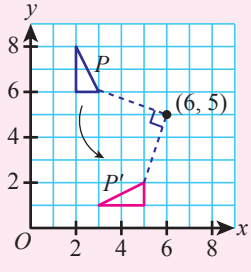
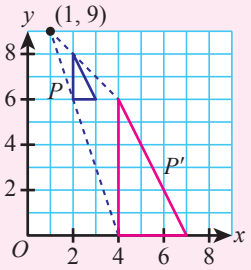
5.3 Combined Transformation

How to determine the image and object of a combined transformation?

Learning Standard

Determine the image and object of a combined transformation.

We have learnt 4 types of transformations as shown in the examples below where P' is the image of P under the transformation stated.

Translation	Reflection	Rotation	Enlargement
Translation $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$	Reflection on line $x = 5$	Rotation of 90° anticlockwise at centre (6, 5)	Enlargement at centre (1, 9) with scale factor 3
			

To determine the object when an image is given, the transformation of translation needs to be performed in the opposite direction. For example, translation $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ becomes $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$. For the transformation of rotation, rotation in clockwise direction will become rotation in anticlockwise direction. For the transformation of enlargement, enlargement with a scale factor $k = 2$ will become reciprocal, that is enlargement with scale factor $k = \frac{1}{2}$.

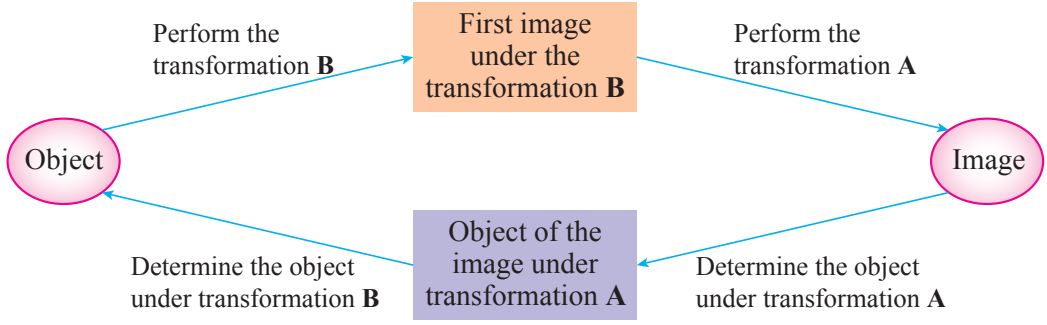
Critical Mind

Why the transformation of reflection does not change like the other three transformations when finding the object if the image is given?

An object can perform more than one transformation and will produce an image based on the transformations involved. In general, the combination between the transformation **A** and transformation **B** can be written as transformation **AB** or transformation **BA** in the order of the desired transformation.

Combined transformation **AB** means transformation **B** followed by transformation **A**.

The diagram below shows the steps to determine the image or object of a combined transformation **AB**.



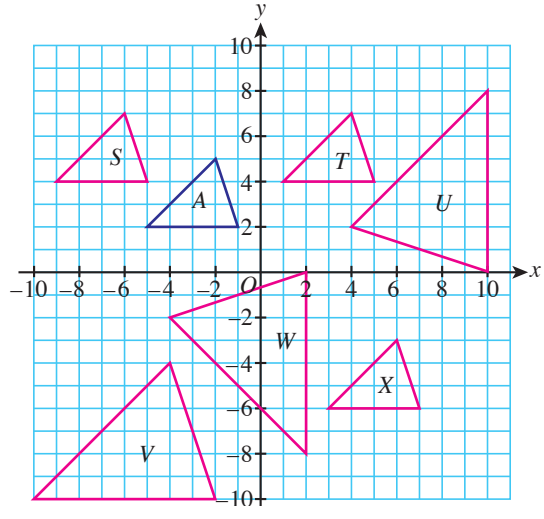
Example 15

The diagram on the right shows several triangles drawn on a Cartesian plane. It is given that transformation

P = translation $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Q = rotation of 90° anticlockwise at centre (3, 4)

R = enlargement at centre (8, 0) with scale factor 2

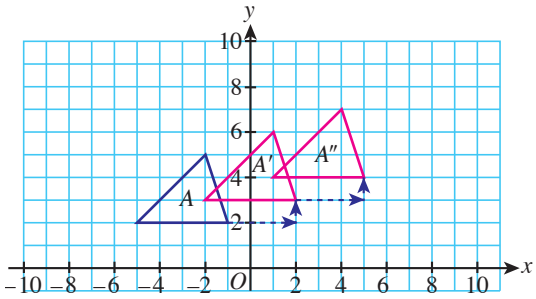


Determine the image of triangle *A* under the combined transformation

(a) $\mathbf{P}^2$ (b) **RQ**

Solution:

(a) Combined transformation $\mathbf{P}^2$ means transformation **P** performed 2 times in a row.

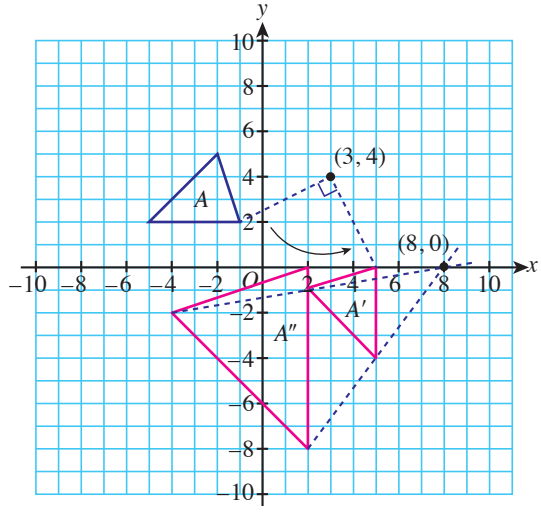


A' is the image of A under transformation **P**.
A'' is the image of A' under transformation **P**.

$$A \xrightarrow{\mathbf{P}} A' \xrightarrow{\mathbf{P}} A''$$

Image of triangle A under the combined transformation $\mathbf{P}^2$ is triangle T.

(b) Combined transformation **RQ** means transformation **Q** followed by transformation **R.**



A' is the image of A under transformation **Q**.
A'' is the image of A' under transformation **R**.

$$A \xrightarrow{\mathbf{Q}} A' \xrightarrow{\mathbf{R}} A''$$

Image of triangle A under the combined transformation **RQ** is triangle W.

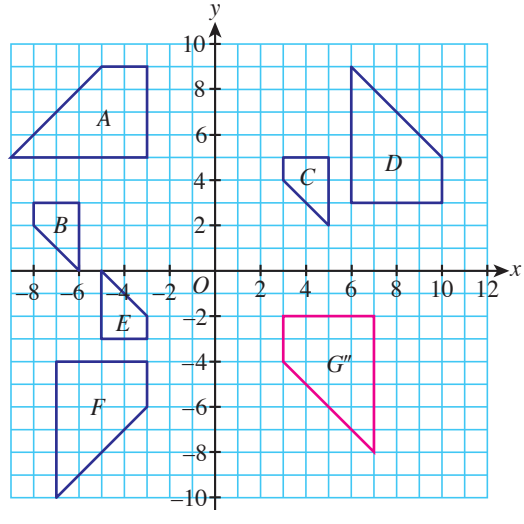
Example 16

In the diagram on the right, trapezium G'' is the image of an object under a combined transformation. It is given that transformation

T = enlargement at centre (1, 4) with scale factor –2
V = rotation of 180° at the origin
W = reflection on line $y = 1$

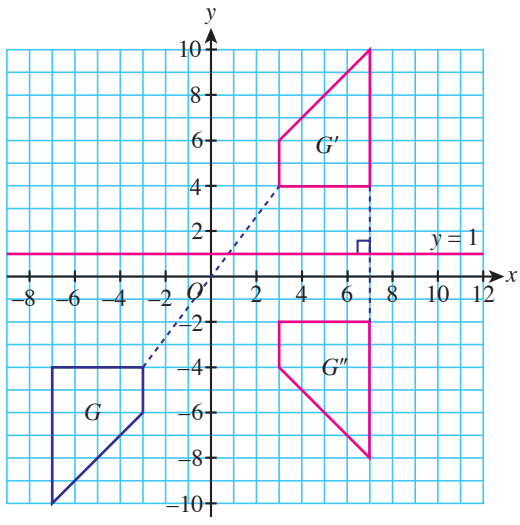
Determine the object of trapezium G'' under the combined transformation

(a) **WV** (b) **VT**



Solution:

(a) To find the object, transformation **W** is followed by transformation **V**.

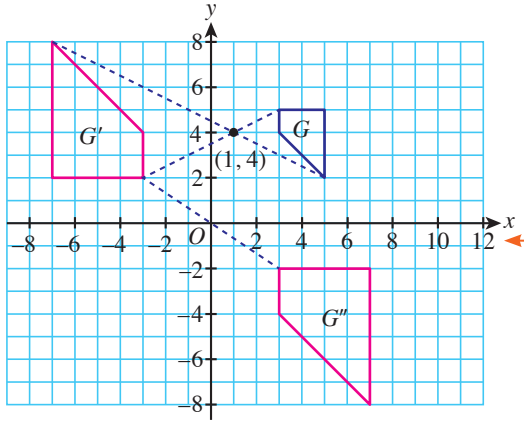


G' is the object of G'' under transformation **W**.
G is the object of G' under transformation **V**.

$$G'' \xrightarrow{\mathbf{W}} G' \xrightarrow{\mathbf{V}} G$$

Object of trapezium G'' under the combined transformation **WV** is trapezium F.

(b) To find the object, transformation **V** is followed by transformation **T**.



G' is the object of G'' under transformation **V**.
G is the object of G' under transformation **T**.

$$G'' \xrightarrow{\mathbf{V}} G' \xrightarrow{\mathbf{T}} G$$

To determine object G under transformation **T**, perform enlargement with scale factor $k = -\frac{1}{2}$ on image G'.

Object of trapezium G'' under the combined transformation **VT** is trapezium C.

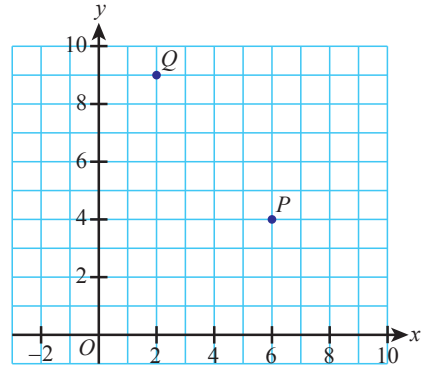
Example 17

The diagram on the right shows the point P and point Q on a Cartesian plane. It is given that transformation

A = translation $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$

B = reflection on line $y = x$

C = rotation of 90° clockwise at centre (5, 8)

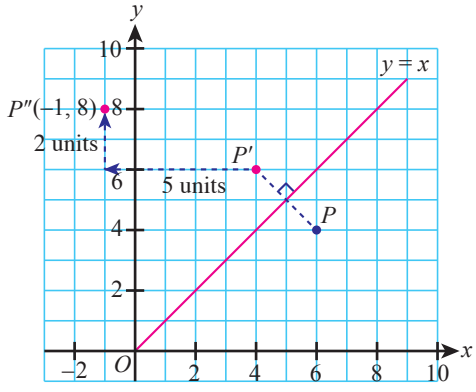


(a) Determine the image of point P under the combined transformation
(i) **AB** (ii) **BC**

(b) It is given that point Q is the image of point K under a transformation **CA**. Determine the point K.

Solution:

(a) (i) Combined transformation **AB** means transformation **B** followed by transformation **A**.

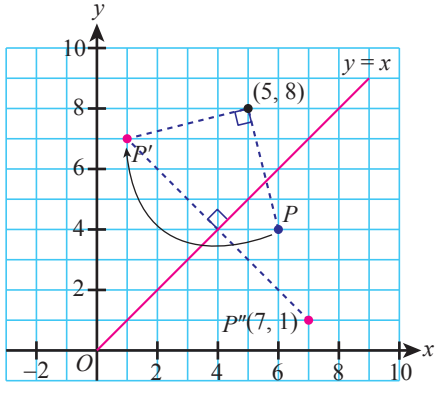


P' is the image of P under transformation **B**.
P'' is the image of P' under transformation **A**.

$$P \xrightarrow{\mathbf{B}} P' \xrightarrow{\mathbf{A}} P''$$

Hence, the image of point P under the combined tranformation **AB** is $P''(-1, 8)$.

(ii) Combined transformation **BC** means transformation **C** followed by transformation **B**.

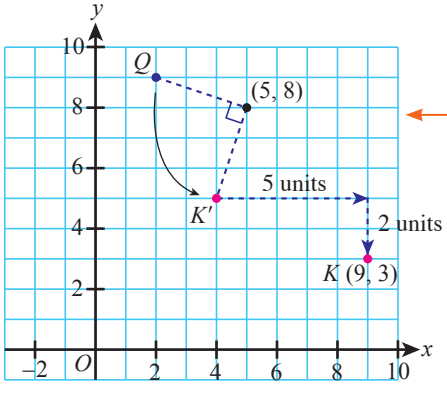


P' is the image of P under transformation **C**.
P'' is the image of P' under transformation **B**.

$$P \xrightarrow{\mathbf{C}} P' \xrightarrow{\mathbf{B}} P''$$

Hence, the image of point P under the combined tranformation **BC** is $P''(7, 1)$.

(b) To find the object, transformation **C** is followed by transformation **A**.



To determine object K' under transformation **C**, perform anticlockwise rotation on image Q.
To determine object K under transformation **A**, perform translation $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ on image K'.

Smart TiPS

When determining the object, the order of transformations is reversed to determine the image under the same combined transformation.

K' is the object of Q under transformation **C**.
K is the object of K' under transformation **A**.

$$Q \xrightarrow{\mathbf{C}} K' \xrightarrow{\mathbf{A}} K$$

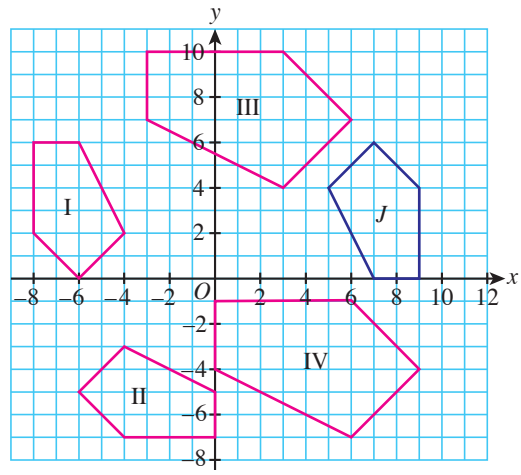
Hence, $K(9, 3)$

Self Practice 5.3a

1. The diagram on the right shows several pentagons drawn on a Cartesian plane. It is given that transformation

A = reflection on line $y = x$
B = rotation of 180° at centre (1, 0)
C = enlargement at centre (6, 7) with scale factor $\frac{3}{2}$

Determine the image of pentagon J under the combined transformation
(a) **AB**
(b) **CA**

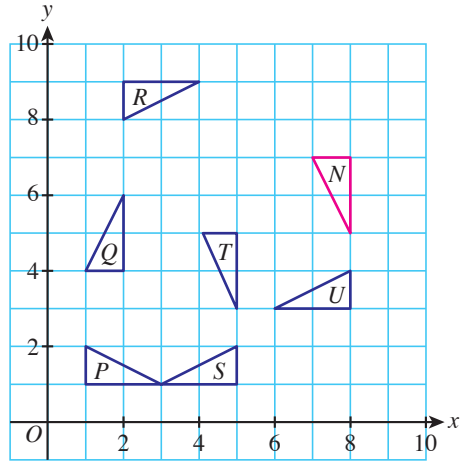


2. In the diagram on the right, the right-angled triangle N is the image of an object under a combined transformation. It is given that transformation

A = translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

B = rotation of 270° clockwise at centre (6, 5)
C = reflection on line $x + y = 6$

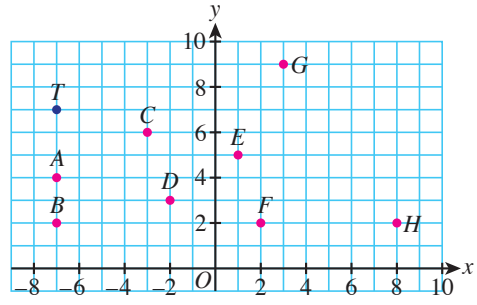
Determine the object of triangle N under the combined transformation
(a) **AC**
(b) **BA**



3. It is given that transformation

U = translation $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$
V = reflection on line $x = 1$
W = rotation of 90° anticlockwise at centre (5, 5)

The diagram on the right shows several points drawn on a Cartesian plane.



(a) Determine the image of point T under the combined transformation
  (i) $\mathbf{U}^2$
  (ii) **WV**
(b) It is given that H is the image of a point under transformation **UV**. Determine the point.

What is the effect of commutative law on combined transformation?

Learning Standard

Make and verify the conjecture about commutative law in combined transformation.

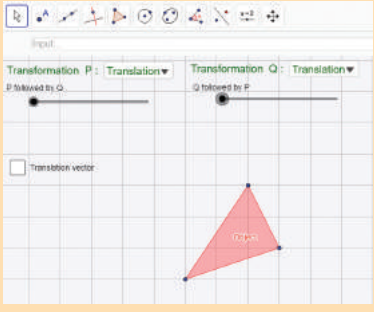
It is given that **P** and **Q** are two different transformations. Is the image under the combined transformation **PQ** the same as the image under the combined transformation **QP**?

MIND MOBILISATION 6 Pairs

Aim: To make and verify the conjecture about commutative law in combined transformation.

Steps:

1. Open the file GGB505 for this activity.



Scan the QR code or visit bit.do/GGB505BI to obtain the GeoGebra file for this activity.

Scan the QR code or visit bit.do/WSChap5ii to obtain the worksheet for this activity.

2. On the menu, select the types of transformation for Transformation P and Transformation Q.
3. Drag the slider 'P followed by Q' and 'Q followed by P' in turns to observe the images produced. Is the image produced by 'P followed by Q' the same as the image produced by 'Q followed by P'?
4. Open the worksheet for this activity. Complete it based on your exploration.

Transformation P	Description of transformation P	Transformation Q	Description of transformation Q	Are images produced by 'P followed by Q' and 'Q followed by P' the same?
Translation		Translation		
Translation		Reflection		

Discussion:

1. State whether each combination of two transformations satisfies the commutative law.
2. What is your conclusion about the commutative law in combined transformation?

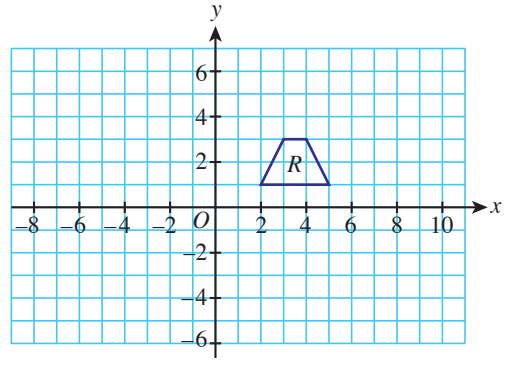
The results of Mind Mobilisation 6 show that most of the images produced by combined transformation **PQ** are not the same as the images produced by combined transformation **QP**.

In general,

- A combined transformation **AB** satisfies the commutative law if the images under the combined transformation **AB** and combined transformation **BA** are the same.
- A combined transformation **AB** does not satisfy the commutative law if the images under the combined transformation **AB** and combined transformation **BA** are not the same.

Example 18

The diagram on the right shows an object on a Cartesian plane. It is given that transformation



A = translation $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

B = reflection on the y-axis

C = rotation of 180° at the origin

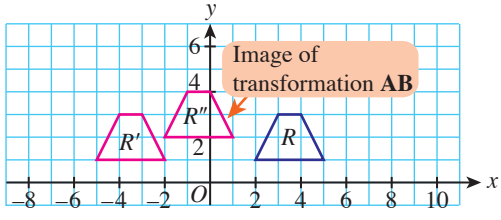
Determine whether each of the following combined transformations satisfies the commutative law.

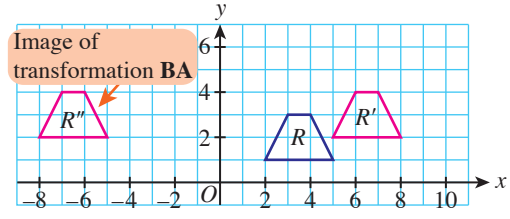
(a) Combined transformation **AB**

(b) Combined transformation **BC**

Solution:

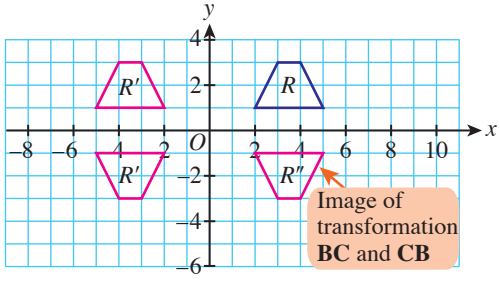
(a)





Images under the combined transformations **AB** and **BA** are not the same, so the combined transformation **AB** does not satisfy the commutative law.

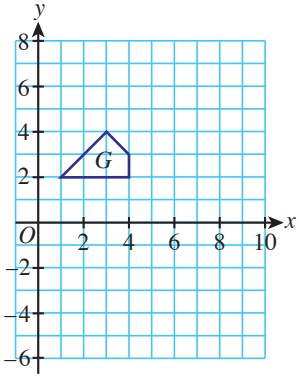
(b)



Images under the combined transformations **BC** and **CB** are the same, so the combined transformation **BC** satisfies the commutative law.

Self Practice 5.3b

1. The diagram on the right shows an object on a Cartesian plane. It is given that transformation



P = enlargement at the origin with scale factor 2

Q = rotation of 90° clockwise at the origin

R = translation $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

Determine whether each of the following combined transformations satisfies the commutative law.

(a) Combined transformation **PQ**

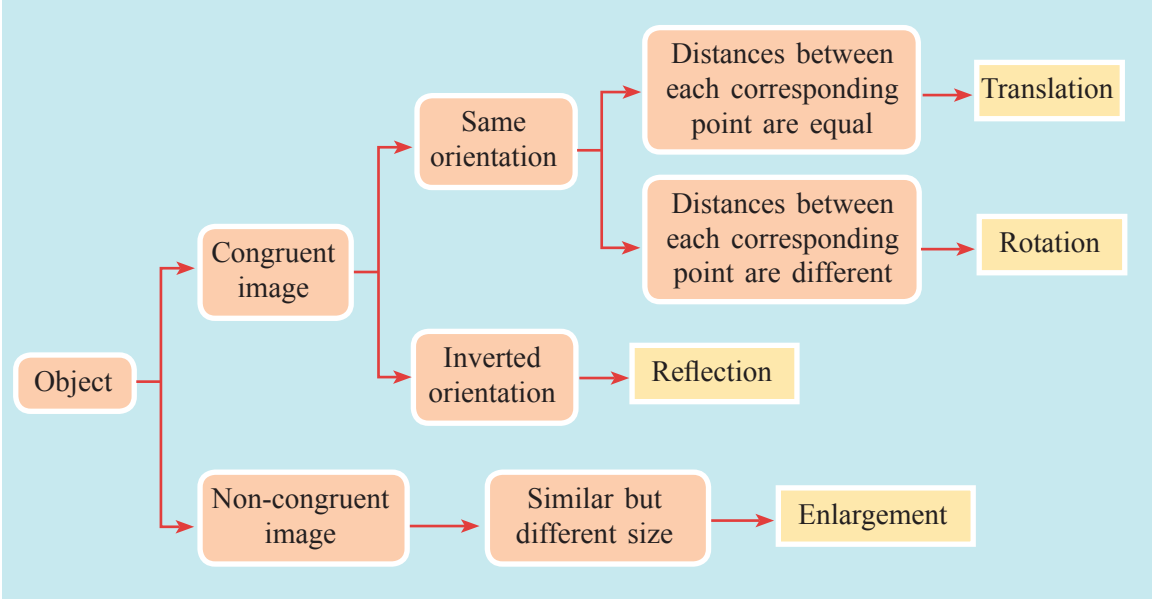
(b) Combined transformation **PR**

How to describe the combined transformation?

Learning Standard

Describe combined transformation.

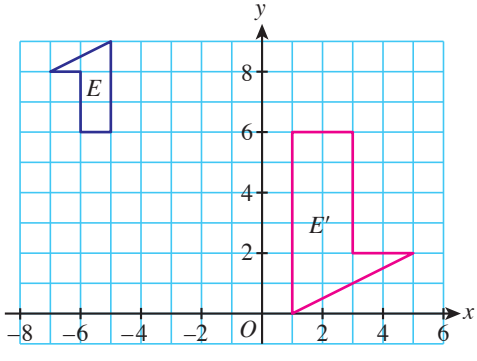
We can describe a transformation based on the properties of the image. The diagram below shows the key points to describe a transformation.



When describing a combined transformation **AB**, we need to describe transformation **B** first, followed by transformation **A**.

Example 19

In the diagram below, *E'* is the image of object *E* under the combined transformation **ST**.

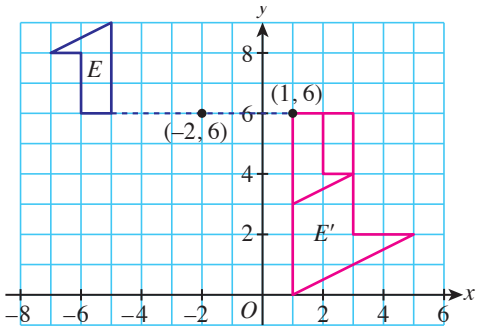


(a) Describe the transformation
  (i) **T**
  (ii) **S**

(b) Hence, describe a single transformation which is equivalent to the combined transformation **ST**.

Solution:

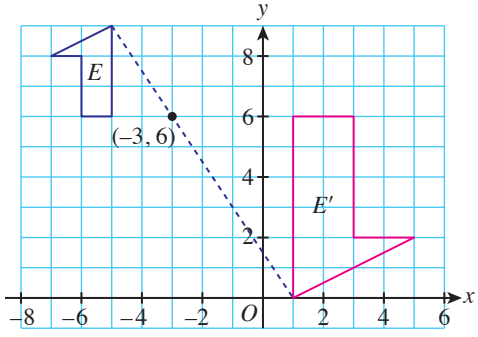
(a)



(i) Transformation **T** is a rotation of 180° at centre (–2, 6).

(ii) Transformation **S** is an enlargement at centre (1, 6) with scale factor 2.

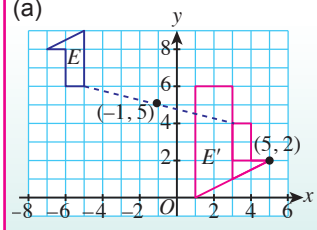
(b)



The single transformation which is equivalent to the combined transformation **ST** is an enlargement at centre (–3, 6) with scale factor –2.

Alternative Method

(a)



(i) Transformation **T** is a rotation of 180° at centre (–1, 5).

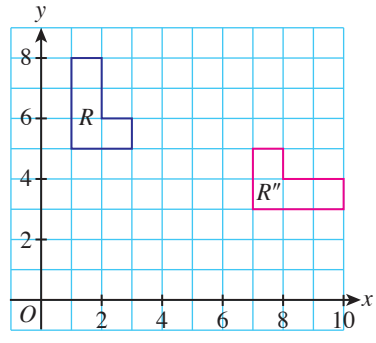
(ii) Transformation **S** is an enlargement at centre (5, 2) with scale factor 2.

Info Bulletin

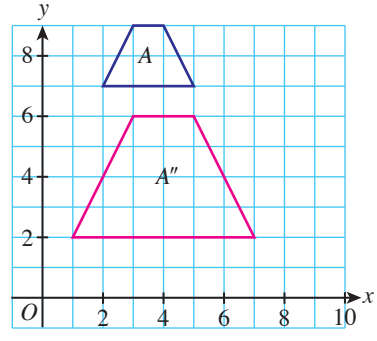
Not all the combined transformations can be described in a single transformation.

Self Practice 5.3c

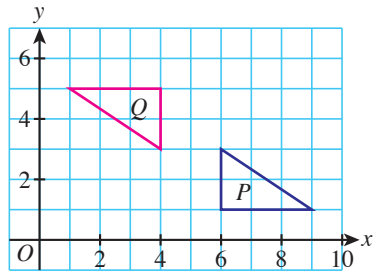
1. In the diagram on the right, R'' is the image of object R under the combined transformation **AB**. Describe transformation **A** and transformation **B**.



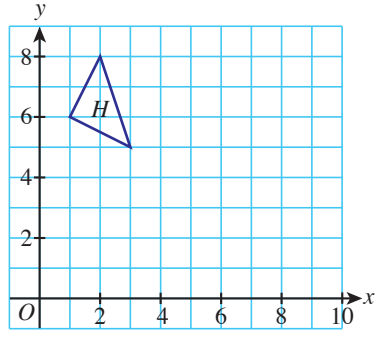
2. In the diagram on the right, A'' is the image of object A under the combined transformation **PQ**. Describe transformation **P** and transformation **Q**.



3. It is given that transformation **A** is a reflection on line $x = 5$ and Q is the image of P under the combined transformation **AB**. Describe
(a) transformation **B**,
(b) a single transformation which is equivalent to the combined transformation **AB**.



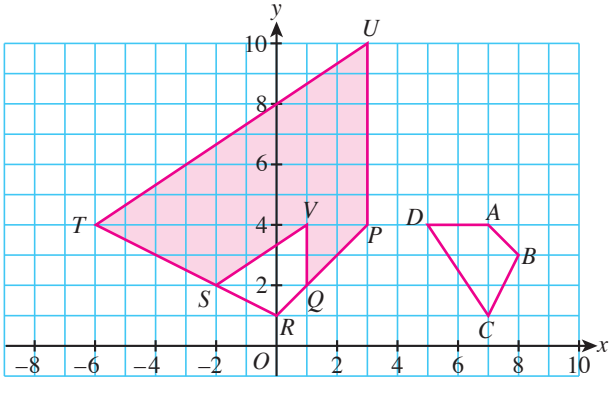
4. It is given that transformation **M** is a rotation of 90° clockwise at centre (4, 4) and transformation **N** is a reflection on line $x = 4$.
(a) Draw the image of object H under the combined transformation **MN**.
(b) Hence, describe a single transformation which is equivalent to the combined transformation **MN**.



How to solve problems involving combined transformation?

Example 20

The diagram below shows the map of a tropical forest. The coloured region is the habitat of a group of wildlife. Quadrilaterals $ABCD$ and $QRSV$ are the wildlife observatories.



The concept of transformation is used to construct the observatory such that $ABCD$ is the image of $PRTU$ under the combined transformation **UV**.
(a) Describe in full, the transformation
(i) **V** (ii) **U**
(b) Given that the area of observatory $ABCD$ is 4.5 m^2, calculate the area, in m^2, of the coloured region.

Learning Standard

Solve problems involving combined transformation.

My Malaysia

According to Wildlife Conservation Act 2010, wildlife means any species of wild animal or wild bird, whether totally protected or protected, vertebrate or invertebrate, live or dead, mature or immature and whether or not may be tamed or bred in captivity.

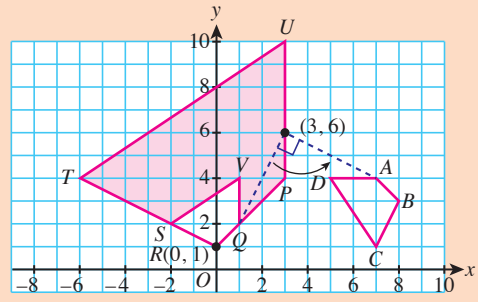
(Source: bit.do/WLConservationAct) (Accessed on 14 September 2020)

Solution:

(a) (i) **V** is an enlargement at centre (0, 1) with scale factor $\frac{1}{3}$.

(ii) **U** is a rotation of 90° anticlockwise at centre (3, 6).

Combined transformation **UV** is an enlargement followed by a rotation. Hence, **V** is the enlargement and **U** is the rotation.



(b) Area of $ABCD = k^2 \times$ Area of $PRTU$

$$4.5 = \left(\frac{1}{3}\right)^2 \times \text{Area of } PRTU$$

Area of $PRTU = 4.5 \times 9$
$= 40.5 \text{ m}^2$

Area of coloured region $= 40.5 - 4.5$
$= 36 \text{ m}^2$

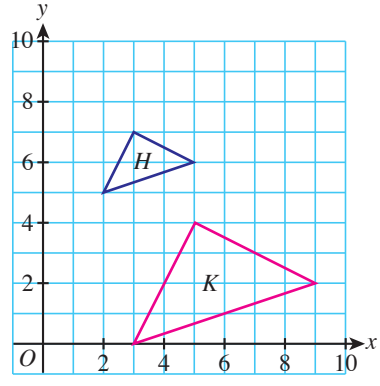
Self Practice 5.3d

1. In the diagram on the right, triangle K is the image of triangle H under the combined transformation **PQ**.

(a) Describe transformation **P** and transformation **Q**.

(b) Describe a single transformation which is equivalent to the transformation **PQ**.

(c) Given that the area of triangle K is 10 unit^2, calculate the area, in unit^2, of triangle H.

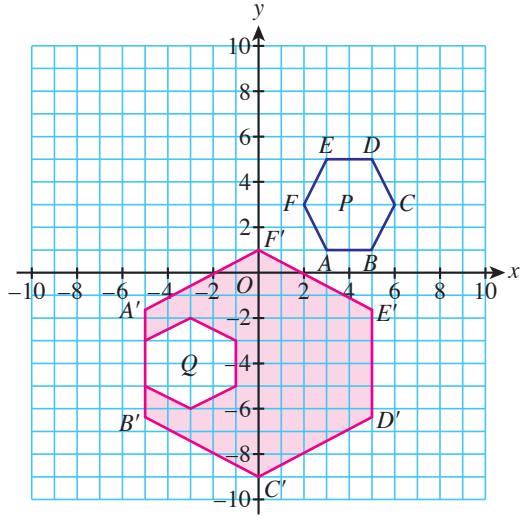


2. The diagram on the right shows a plan of a water theme park. The coloured region is for water tunnels. The hexagonal regions P and Q are food courts. During the construction of the water theme park, the architects used the concept of transformation to design the plan such that Q is the image of P under a reflection and the hexagon $A'B'C'D'E'F'$ is the image of P under the combined transformation **UV**.

(a) Determine the axis of reflection.

(b) Describe transformation **U**.

(c) Given that the area of the food court P is 60 m^2, calculate the area, in m^2, of the coloured region.



5.4 Tessellation

What is the meaning of tessellation?

Learning Standard

Explain the meaning of tessellation.

Beehives have a unique inner structure. This structure can provide the maximum space to keep honey. What are the characteristics of the structure of a beehive and how is this structure constructed?

The structure of a beehive is a **tessellation**.

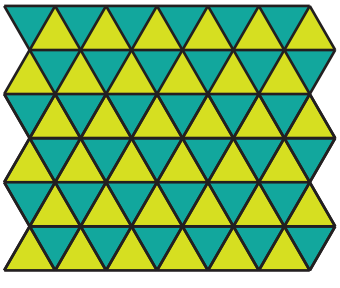
A **tessellation** is a pattern of recurring shapes that fills a plane without leaving empty spaces or overlapping.

Mathematics is fun!

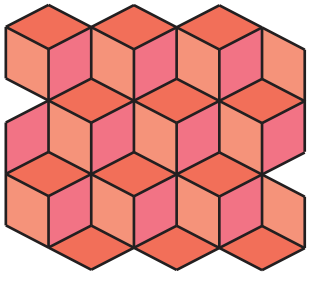
Test Your Understanding About Tessellation
Scan the QR code or visit bit.do/GGB506BI. Drag the points on the screen until a tessellation is created. Print your work and show to your teacher.

The diagram below shows some examples of tessellation.

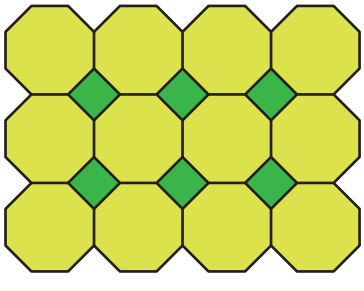
(i) Tessellation consisting of equilateral triangles



(ii) Tessellation consisting of rhombuses



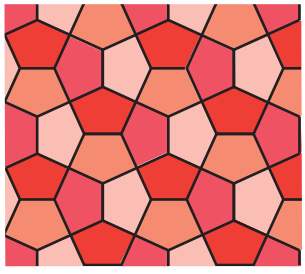
(iii) Tessellation consisting of a combination of squares and regular octagons



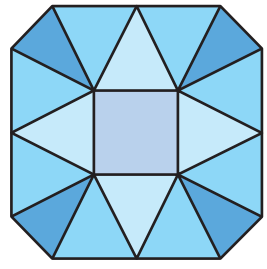
Example 21

Determine whether each of the following patterns is a tessellation.

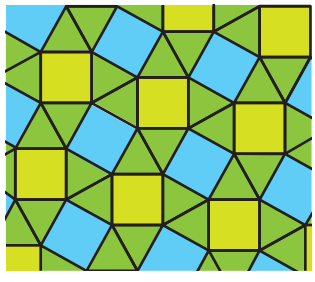
(a)



(b)



(c)



(d)

Solution:

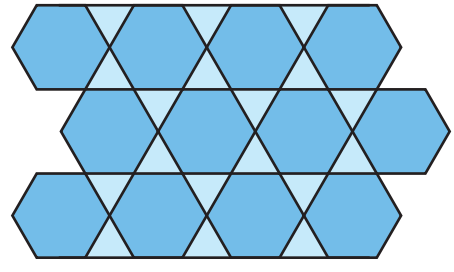
(a) A tessellation consisting of pentagons only.
(b) Not a tessellation because it has a shape which does not recur, which is a square.
(c) A tessellation consisting of combination of equilateral triangles and squares.
(d) A tessellation consisting of the recurring patterns without overlapping.

Self Practice 5.4a

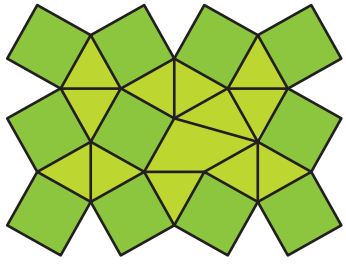
1. Determine whether each of the following patterns is a tessellation.

(a)

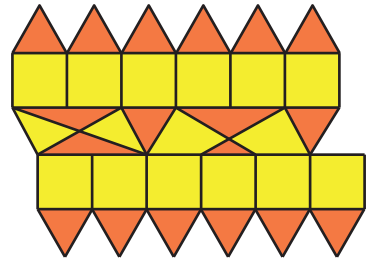
(b)



(c)



(d)



How to design the tessellation?

Learning Standard

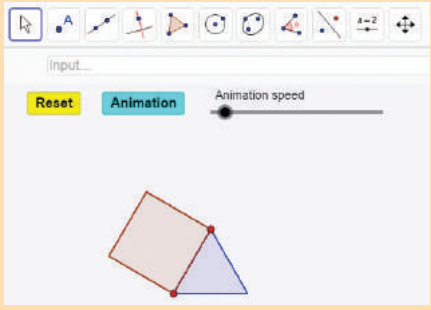
Design tessellation involving isometric transformation.

MIND MOBILISATION 7 Pairs

Aim: To design tessellation by using isometric transformation

Steps:

1. Open the file GGB507 for this activity.



Scan the QR code or visit bit.do/GGB507BI to obtain the GeoGebra file for this activity.

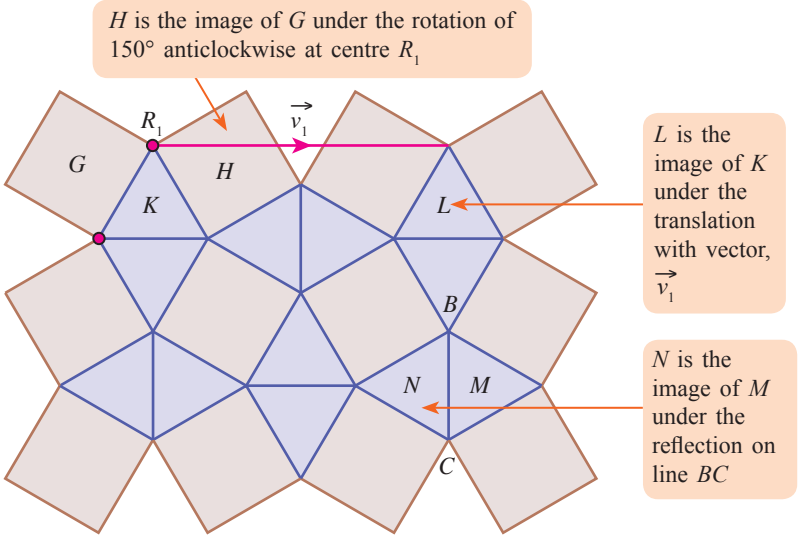
2. Click 'Animation'.
3. Observe the transformation involved in producing a tessellation.
4. Drag slider 'Animation speed' at an appropriate speed.
5. Click 'Reset' if you want to make another observation.

Discussion:

1. How many types of transformations are used to design a tessellation in this activity? State all the transformations.
2. Discuss how the combined transformation is used to design a tessellation.

The results of Mind Mobilisation 7 show that we can design a tessellation by using isometric transformations such as translation, reflection and rotation.

The following shows some steps involved in producing the tessellation in the activity above.



MEMORY BOX

Isometric transformation is the transformation that produces an image that is congruent with an object.

Example 22

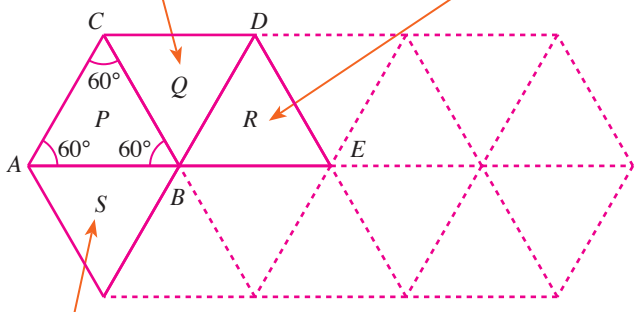
The diagram below shows a tessellation consisting of equilateral triangles which are produced by isometric transformation. State the transformation involved to produce

(a) shape Q from shape P
(b) shape R from shape Q
(c) shape S from shape P

Solution:

Q is the image of P under
- the reflection on line BC or
- the rotation of 60° clockwise at centre B

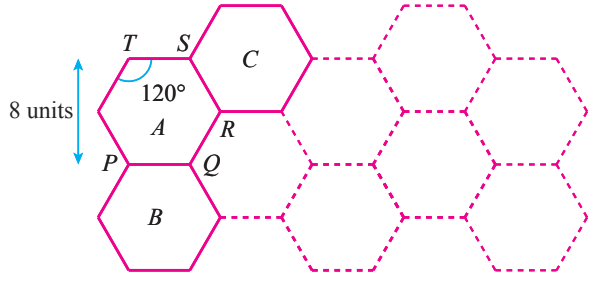
R is the image of Q under
- the reflection on line BD or
- the rotation of 60° anticlockwise at centre D



S is the image of P under
- the reflection on line AB or
- the rotation of 60° clockwise at centre A

Self Practice 5.4b

1. The diagram below shows a tessellation consisting of regular hexagons which are produced by isometric transformation.

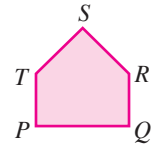


State the transformation involved to produce

(a) shape B from shape A
(b) shape C from shape A

Interactive Platform

The diagram below shows a pentagon $PQRST$.



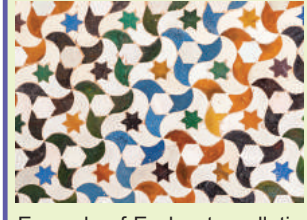
By using the pentagon, discuss in groups on how to design several different tessellations by using combined transformations.

Info Bulletin

R is also the image of P under the translation $\begin{pmatrix} d \\ 0 \end{pmatrix}$, such that d = distance of AB

Info Bulletin

Escher Tessellation



Example of Escher tessellation

Maurits Cornelis Escher (1898-1972) was a Dutch graphic artist. He designed a lot of artworks based on the concept of mathematics. He introduced a unique and attractive way to produce the pattern of tessellation which is known as Escher tessellation. An Escher tessellation is a tessellation in which its shapes are easily identified in real life.

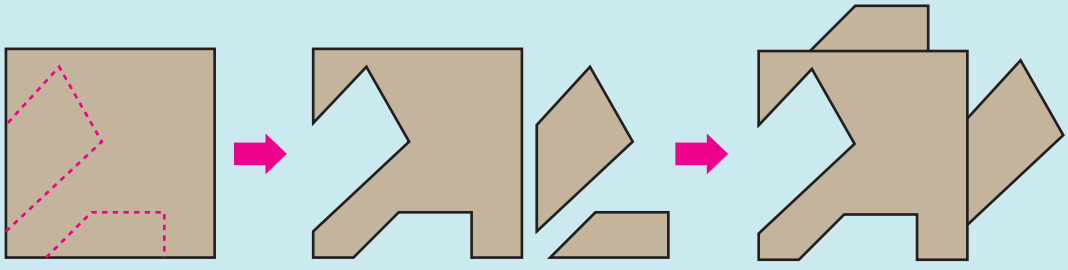
PROJECT

Aim: To design an Escher tessellation

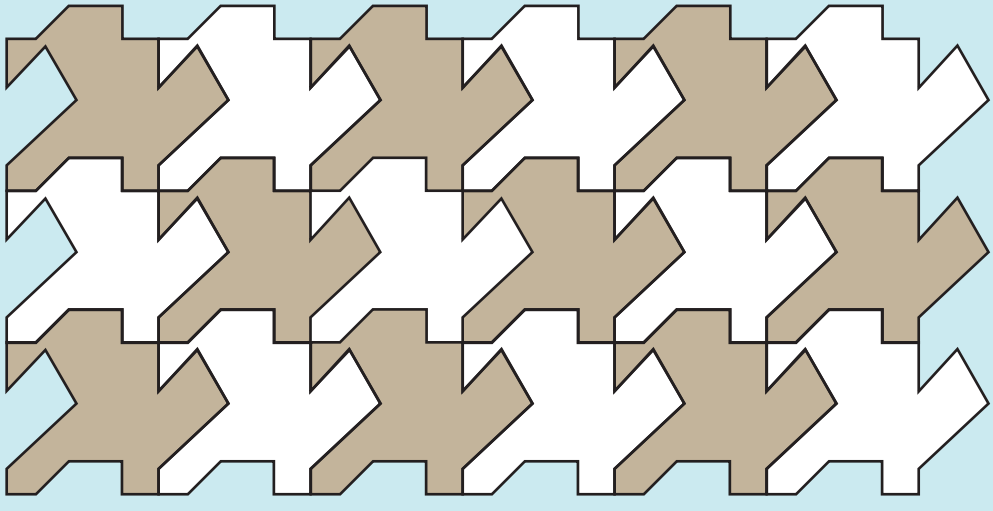
Materials: Cardboard, coloured papers, scissors, glue

Steps:

1. Carry out this project in groups.
2. Cut the provided coloured papers into 18 squares of equal size.
3. Choose two sides on a square, draw and cut out two different patterns. Then, paste the remaining part on the cardboard and the two cut patterns are attached on the other two sides of the remaining part as shown in the diagram below.



4. Repeat step **3** to produce another 17 identical patterns.
5. Arrange and paste all the shapes produced on the cardboard to produce the Escher tessellation as shown in the diagram below. What is the characteristic that differentiates the usual tessellation with the Escher tessellation?



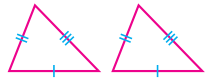
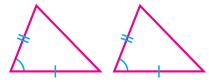
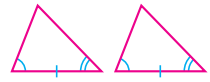
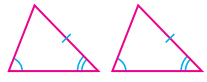
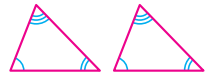
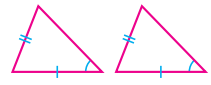
6. Repeat this project to design several different Escher tessellations.
7. Display the design of your group in the class.
8. Present
   (a) the transformation used in the design of your group.
   (b) your conclusion about Escher tessellation.

Summary Arena

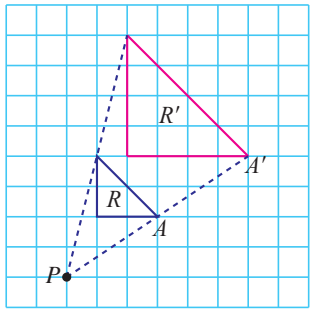
Congruent shapes

- The corresponding sides and angles are of the same sizes.

Triangle congruency

• Side-Side-Side (SSS) 	• Side-Angle-Side (SAS) 	• Angle-Side-Angle (ASA) 
• Angle-Angle-Side (AAS) 	• Angle-Angle-Angle (AAA)  Area must be equal	• Side-Side-Angle (SSA)  Area must be equal

Enlargement



- Object and image are similar
- Scale factor, $k = \frac{PA'}{PA}$
- Area of image $= k^2 \times$ Area of object

Combined transformation

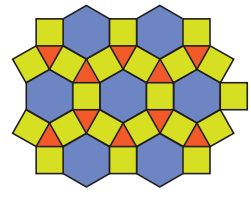
- Combined transformation **AB** means transformation **B** followed by transformation **A**

To determine image from object under transformation **AB**, the order is transformation **B** followed by transformation **A**.

To determine object from image, the order of transformations is reversed to determine the image under the same combined transformation.

Tessellation

- Tessellation is a pattern of recurring shapes that fills a plane without leaving empty spaces or overlapping
- Tessellation can be designed from an isometric transformation



Reflection

At the end of this chapter, I can

		
differentiate between congruent and non-congruent shapes based on sides and angles.		
make and verify the conjecture of triangle congruency based on sides and angles.		
solve problems involving congruency.		
explain the meaning of similarity of geometric objects.		
make a connection between similarity and enlargement, hence describe enlargement using various representations.		
determine the image and object of an enlargement.		
make and verify conjecture on the relation between area of the image and area of the object of an enlargement.		
solve problems involving enlargement.		
determine the image and object of a combined transformation.		
make and verify the conjecture about commutative law in combined transformation.		
describe combined transformation.		
solve problems involving combined transformation.		
explain the meaning of tessellation.		
design tessellation involving isometric transformation.		

MINI PROJECT

Batik is a part of Malaysian cultural heritage. The design of *batik Malaysia* is world-renowned. Use your creativity to design a colourful batik pattern by using the Escher tessellation. You may use the dynamic geometry software or concrete materials to produce the batik pattern. Then, display your work in the class.

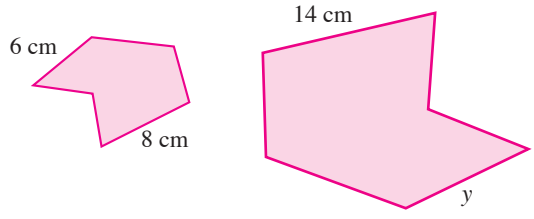
Extensive Practice

Scan the QR code or visit bit.do/QuizE05 for interactive quiz

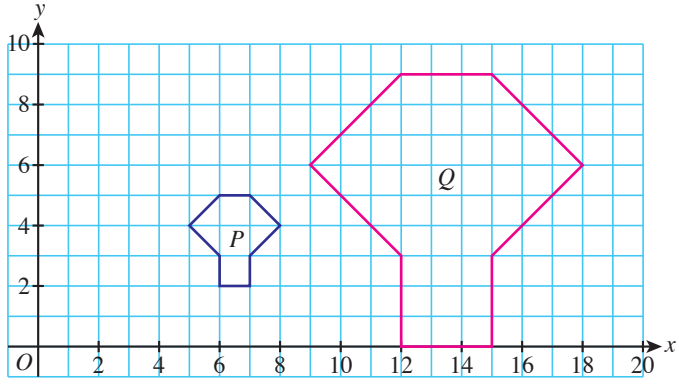
UNDERSTAND

1. The diagram on the right shows two congruent trapeziums with a pair of sides of equal length. Calculate the value of x.

2. The diagram on the right shows two similar geometric objects. Calculate the value of y.



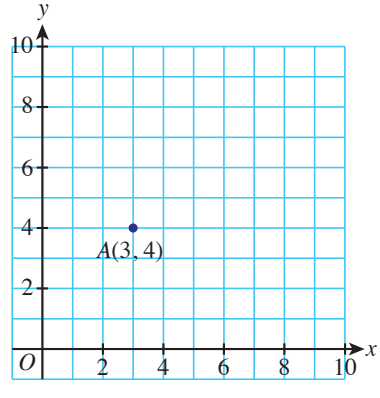
3. The diagram below shows that Q is the image of P under an enlargement.



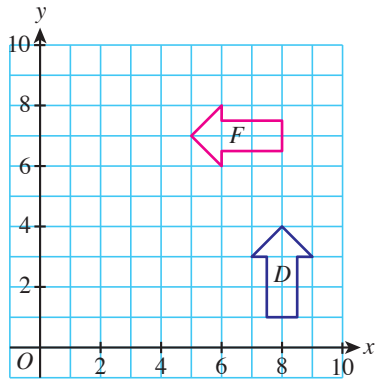
Describe the enlargement.

4. It is given that the areas of shape A and shape B are 18 cm^2 and 32 cm^2 respectively. If shape B is the image of shape A under an enlargement, determine the scale factor of the enlargement.

5. The diagram on the right shows that point A lies on a Cartesian plane. It is given that transformation **R** is translation $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and transformation **S** is a reflection on line $y = 7$. State the coordinates of the image of point A under the combined transformation **RS.**



6. The diagram on the right shows that F is the image of D under the combined transformation **GH**. It is given that transformation **H** is a reflection on line $x = 6$. Describe transformation **G**.

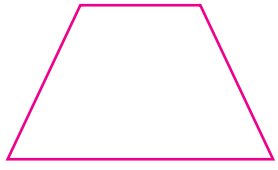


MASTERY

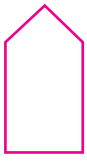
7. Based on each of the following polygons, design a tessellation using isometric transformation.

(a) isosceles triangle

(b) trapezium

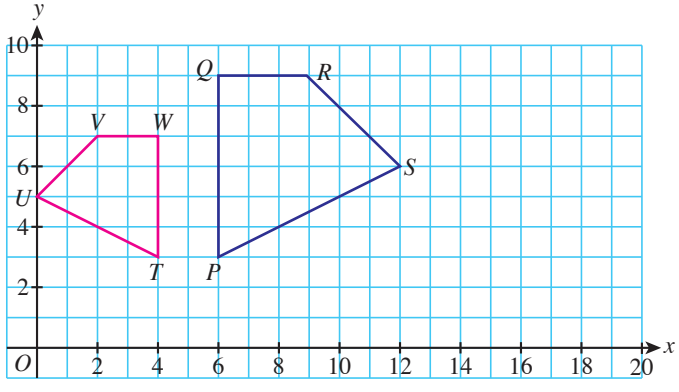


(c) pentagon



(d) hexagon

8. The diagram below shows two quadrilaterals $PQRS$ and $TUVW$ drawn on a Cartesian plane.

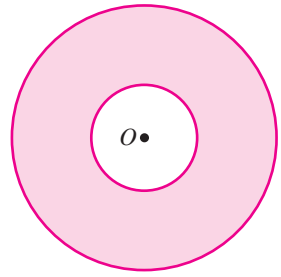


(a) Quadrilateral $TUVW$ is the image of quadrilateral $PQRS$ under the combined transformation **MN**. Describe
   (i) transformation **N**,
   (ii) transformation **M**.

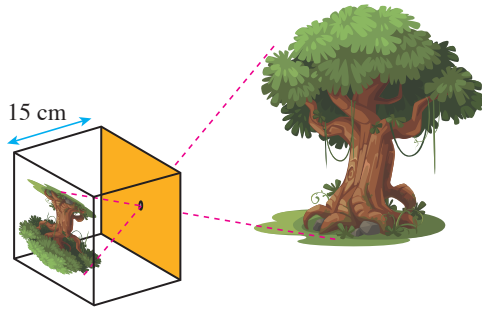
(b) Given that the area of quadrilateral $TUVW$ is 10 unit^2, calculate the area, in unit^2, of the quadrilateral $PQRS$.

CHALLENGE

9. The diagram on the right shows two circles with the same centre O. The large circle is the image of the small circle under an enlargement at centre O. It is given that the radius of the small circle is 3 cm and the area of the coloured region is 47.25π cm^2. Calculate the scale factor of the enlargement.



10. Zurina built a pinhole camera to observe the image of a tree as shown in the diagram below.

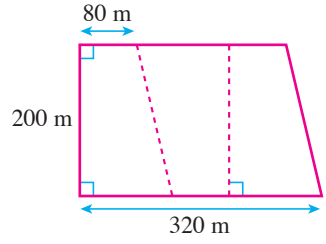


It is given that the scale factor of the image produced is $-\frac{1}{80}$. What is the horizontal distance, in cm, between the tree and its image?

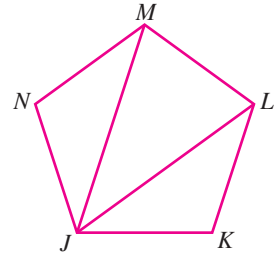
11. Madam Noriah has a trapezium-shaped piece of land. She divided her land into three congruent sections as shown in the diagram on the right. Calculate

(a) the perimeter, in m, of the land,

(b) the area, in m^2, of each section of the land.



12. The diagram on the right shows a regular pentagon $JKLMN$. It is given that the triangle JNM is the image of triangle JKL under a combined transformation **XY**. Describe transformation **X** and transformation **Y**.



EXPLORING MATHEMATICS

Diagram (a) below shows several shapes drawn on a square grid.

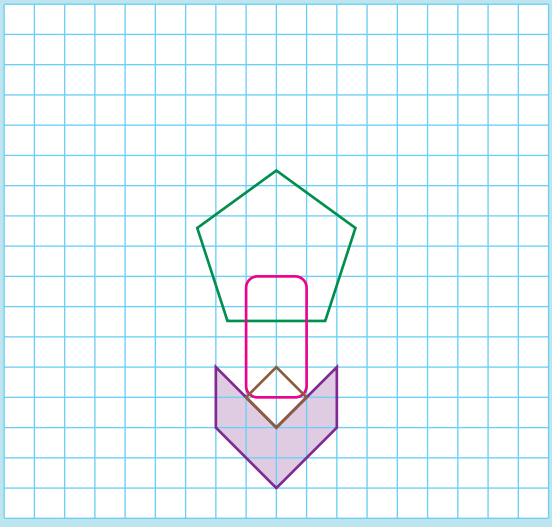
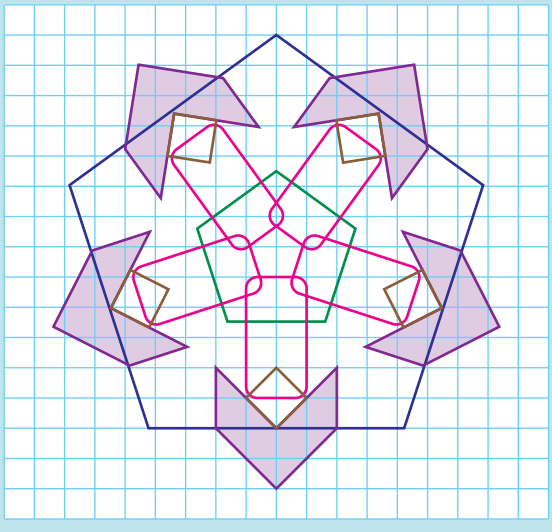


Diagram (b) below shows a logo designed by applying the transformation to the shapes in Diagram (a) and it is drawn on the square grid.



Based on your observation and discussion in groups, describe the transformations that have been used to design the logo.

Then, by choosing a combination of polygons, design a logo for an association or club in your school by applying transformations to the combination of polygons. Present the logo designed based on the following aspects:

(i) The meaning represented by each part or colour of the logo.
(ii) The appropriate motto.
(iii) The combined transformations used.

CHAPTER 6

Ratios and Graphs of Trigonometric Functions

What will you learn?

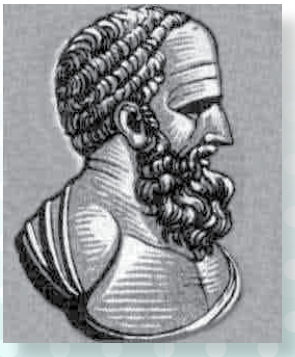
- The Value of Sine, Cosine and Tangent for Angle θ, $0° \leqslant \theta \leqslant 360°$
- The Graphs of Sine, Cosine and Tangent Functions

Why study this chapter?

Engineers use graphs of trigonometric functions in the construction of concert halls to measure sound strength so that the sound is clearly heard. Geologists also use graphs of trigonometric functions to help them understand the formation of periodic pattern of earthquakes and waves.

Do you know?

Hipparchus from Nicaea was an astronomer and mathematician known as the Father of Trigonometry for his contribution to trigonometry. He developed a trigonometric table in his attempt to understand the movement of stars and moons.



For more information:

bit.do/DoYouKnowChap6

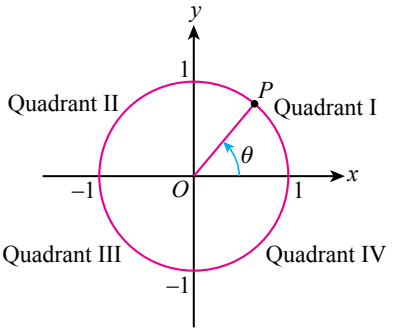
WORD BANK

unit circle	*bulatan unit*
trigonometric function	*fungsi trigonometri*
cosine	*kosinus*
sine	*sinus*
corresponding reference angle	*sudut rujukan sepadan*
quadrant	*sukuan*
tangent	*tangen*

Langkawi Island, known as the Jewel of Kedah, is one of the tidal stations in Malaysia. A tidal station is built to collect tidal data around Langkawi Island. This data can be used to detect tsunami waves. As tides occur periodically, such tidal data can be modelled with graphs of trigonometric functions.

6.1 The Value of Sine, Cosine and Tangent for Angle θ, $0° \leqslant \theta \leqslant 360°$

The diagram on the right shows a unit circle. A **unit circle** is a circle that has a radius of 1 unit and is centred on the origin. The x-axis and y-axis divide the unit circle into 4 equal quadrants, namely quadrant I, quadrant II, quadrant III and quadrant IV.



It is given that P is a point that moves along the circumference of the unit circle and θ is the angle formed by the radius of the unit circle, OP, from the positive x-axis in an anticlockwise direction. It is found that

(a) point P is in quadrant I when $0° < \theta < 90°$,
(b) point P is in quadrant II when $90° < \theta < 180°$,
(c) point P is in quadrant III when $180° < \theta < 270°$,
(d) point P is in quadrant IV when $270° < \theta < 360°$.

What is the relationship between the function of sine, cosine and tangent for angles in quadrants II, III and IV with the corresponding reference angle?

Learning Standard

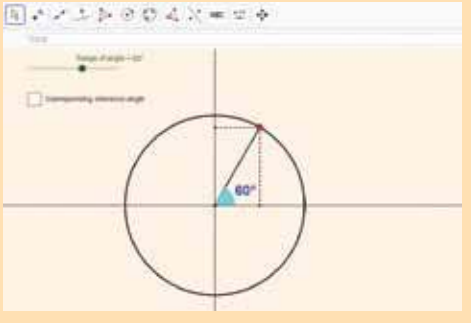
Make and verify conjecture about the value of sine, cosine and tangent for angles in quadrants II, III and IV with the corresponding reference angle.

MIND MOBILISATION 1 Pairs

Aim: To explore the relationship between the function of sine, cosine and tangent for angles in quadrants II, III and IV with the corresponding reference angle.

Steps:

1. Open the file GGB601 for this activity.



Scan the QR code or visit bit.do/GGB601BI to obtain the GeoGebra file for this activity.

2. Drag the red point to quadrants II, III and IV. Observe the blue angle.
   [The slider 'Range of angle' can be dragged to change the displayed angle.]
3. Click 'Corresponding reference angle'.
4. Drag the red point and observe the corresponding reference angles in quadrants II, III and IV.

5. Write the relationship between the corresponding reference angle, α, with the blue angle, θ, in each quadrant.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\alpha =$	$\alpha =$	$\alpha =$	$\alpha =$

6. Drag the red point and choose any blue angles in quadrants II, III and IV to complete the table below. Use a scientific calculator to do the calculations.

Quadrant ☐ :

Blue angle, $\theta =$ ☐ Corresponding reference angle, $\alpha =$ ☐

$\sin\theta$	$\sin\alpha$	$\cos\theta$	$\cos\alpha$	$\tan\theta$	$\tan\alpha$

7. Based on the table in step **6**, compare the values of sine, cosine and tangent of blue angle, θ, with the corresponding reference angle, α. Complete each of the following with positive or negative sign.

Quadrant II

$\sin\theta =$ ☐ $\sin\alpha$

$\cos\theta =$ ☐ $\cos\alpha$

$\tan\theta =$ ☐ $\tan\alpha$

Quadrant III

$\sin\theta =$ ☐ $\sin\alpha$

$\cos\theta =$ ☐ $\cos\alpha$

$\tan\theta =$ ☐ $\tan\alpha$

Quadrant IV

$\sin\theta =$ ☐ $\sin\alpha$

$\cos\theta =$ ☐ $\cos\alpha$

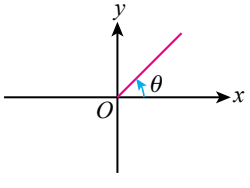
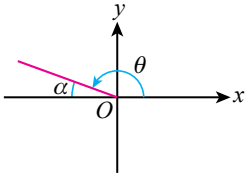
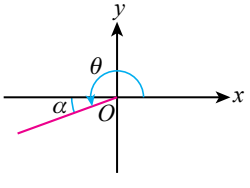
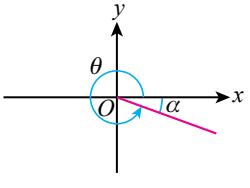
$\tan\theta =$ ☐ $\tan\alpha$

Discussion:

1. What are the characteristics of the corresponding reference angles and their relationship with angles in quadrants II, III and IV?
2. What is your conclusion about the relationship of the function of sine, cosine and tangent for angles in quadrants II, III and IV with the corresponding reference angle?

The results of Mind Mobilisation 1 show that;

(a) The **corresponding reference angle**, α, is always less than 90°. Angles in quadrants II, III and IV have corresponding reference angles, α. The angle in quadrant I itself is the corresponding reference angle, $\alpha = \theta$.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
			
$\alpha = \theta$	$\alpha = 180° - \theta$	$\alpha = \theta - 180°$	$\alpha = 360° - \theta$

The reference angles in quadrants II, III and IV are the corresponding angles in quadrant I.

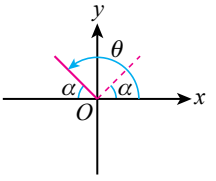
(b) The relationship between the function of sine, cosine and tangent for angles in quadrants II, III and IV with the corresponding reference angle can be summarised as follows:

Quadrant II

$\sin\theta = +\sin\alpha = +\sin(180° - \theta)$

$\cos\theta = -\cos\alpha = -\cos(180° - \theta)$

$\tan\theta = -\tan\alpha = -\tan(180° - \theta)$

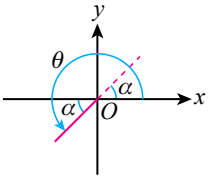


Quadrant III

$\sin\theta = -\sin\alpha = -\sin(\theta - 180°)$

$\cos\theta = -\cos\alpha = -\cos(\theta - 180°)$

$\tan\theta = +\tan\alpha = +\tan(\theta - 180°)$

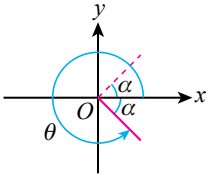


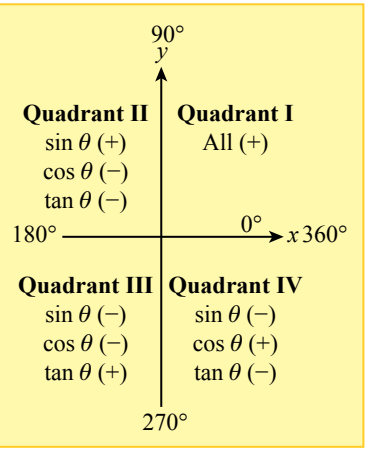
Quadrant IV

$\sin\theta = -\sin\alpha = -\sin(360° - \theta)$

$\cos\theta = +\cos\alpha = +\cos(360° - \theta)$

$\tan\theta = -\tan\alpha = -\tan(360° - \theta)$





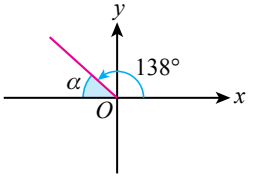
Example 1

Determine the quadrant and the corresponding reference angle for each of the following.

(a) 138° (b) 239° (c) 312°

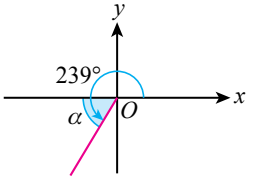
Solution:

(a)



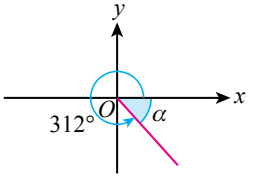
138° is located in quadrant II.
Corresponding reference angle, α
$= 180° - 138°$
$= 42°$

(b)



239° is located in quadrant III.
Corresponding reference angle, α
$= 239° - 180°$
$= 59°$

(c)



312° is located in quadrant IV.
Corresponding reference angle, α
$= 360° - 312°$
$= 48°$

Smart TIPS

Corresponding reference angle is an acute angle.

Critical Mind

To obtain the corresponding reference angle, α, why is the formula $\theta - 180°$ used in Example 1(b)?

Critical Mind

Calculate the corresponding reference angle if the given angle is 498°.

Example 2

State the relationship between each of the following trigonometric functions with its corresponding reference angle.

(a) sin 167° (b) cos 258° (c) tan 349°

Solution:

only sin θ is positive

(a) Angle 167° is located in quadrant II.

$\sin \theta = \sin (180° - \theta)$

$\sin 167° = \sin (180° - 167°)$

$\sin 167° = \sin 13°$

only tan θ is positive

(b) Angle 258° is located in quadrant III.

$\cos \theta = -\cos (\theta - 180°)$

$\cos 258° = -\cos (258° - 180°)$

$\cos 258° = -\cos 78°$

only cos θ is positive

(c) Angle 349° is located in quadrant IV.

$\tan \theta = -\tan (360° - \theta)$

$\tan 349° = -\tan (360° - 349°)$

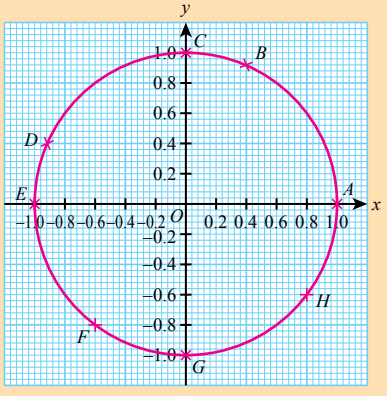
$\tan 349° = -\tan 11°$

MIND MOBILISATION 2 Pairs

Aim: To explore the relationship between the values of sine, cosine and tangent with the values of x-coordinate and y-coordinate for angles in quadrants II, III and IV in a unit circle.

Steps:

1. Draw the x-axis and the y-axis with the origin O on a graph paper and a circle centred at O with a radius of 1 unit as shown in the diagram on the right.
2. Plot the points as the diagram on the right.
3. Copy and complete the table as follows.



Point	x-coordinate	y-coordinate	$\frac{y\text{-coordinate}}{x\text{-coordinate}}$
A			
B			

4. It is given that θ is the angle formed by the radius of a unit circle from the positive x-axis in an anticlockwise direction. Measure the angle, θ from the drawn graph and complete the table as follows.

Radius	Angle θ	sin θ	cos θ	tan θ
OA				
OB				

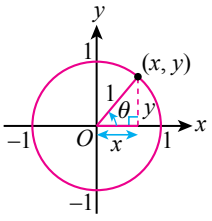
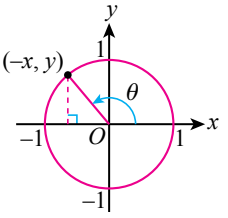
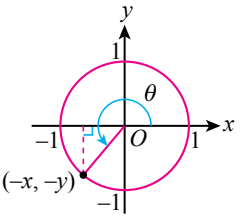
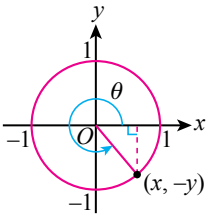
Discussion:

1. What is the relationship between the values of x-coordinate, y-coordinate and the ratio of y-coordinate to x-coordinate with the values of sin θ, cos θ and tan θ?

The results of Mind Mobilisation 2 show that;

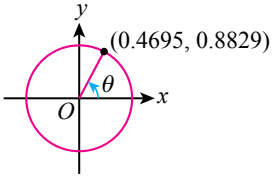
$\sin \theta = y\text{-coordinate}$ $\cos \theta = x\text{-coordinate}$ $\tan \theta = \frac{y\text{-coordinate}}{x\text{-coordinate}}$

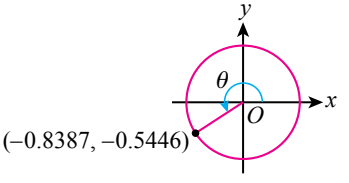
In general,

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
			
$\sin\theta = +y$ $\cos\theta = +x$ $\tan\theta = +\frac{y}{x}$	$\sin\theta = +y$ $\cos\theta = -x$ $\tan\theta = -\frac{y}{x}$	$\sin\theta = -y$ $\cos\theta = -x$ $\tan\theta = +\frac{y}{x}$	$\sin\theta = -y$ $\cos\theta = +x$ $\tan\theta = -\frac{y}{x}$

Example 3

Each of the following diagrams shows a unit circle and angle θ. Determine the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

(a) 

(b) 

Solution:

(a) $\sin\theta = 0.8829$
$\cos\theta = 0.4695$
$\tan\theta = \frac{0.8829}{0.4695}$
$= 1.8805$

(b) $\sin\theta = -0.5446$
$\cos\theta = -0.8387$
$\tan\theta = \frac{-0.5446}{-0.8387}$
$= 0.6493$

Self Practice 6.1a

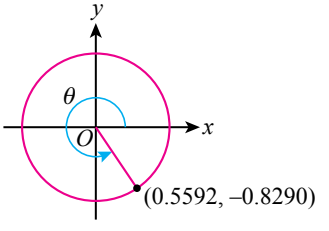
1. Determine the value of corresponding reference angle for each of the following.

(a) 97° (b) 189° (c) 278° (d) 164.2°
(e) 253.6° (f) 305.7° (g) 128°53′ (h) 215°42′

2. State the relationship between each of the following trigonometric functions with its corresponding reference angle.

(a) sin 101° (b) cos 194° (c) tan 246°
(d) tan 294.5° (e) sin 339.8° (f) cos 112.3°
(g) cos 287°45′ (h) tan 96°31′ (i) sin 203°26′

3. The diagram on the right shows a unit circle and angle θ. Determine the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.



How to determine the value of sine, cosine and tangent for angles in quadrants II, III and IV?

Determine the corresponding reference angle. → Determine positive or negative sign for sine, cosine and tangent for angles in quadrants II, III and IV.

Learning Standard

Determine the value of sine, cosine and tangent for angles in quadrants II, III and IV based on the corresponding reference angle.

Example 4

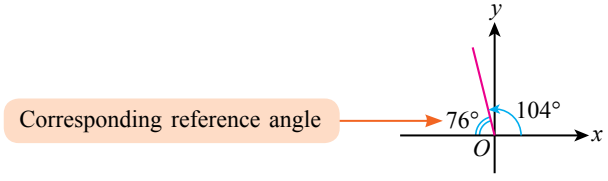
Determine the value for each of the following based on the corresponding reference angle.

(a) sin 104° (b) cos 208.2° (c) tan 318°17′

Solution:

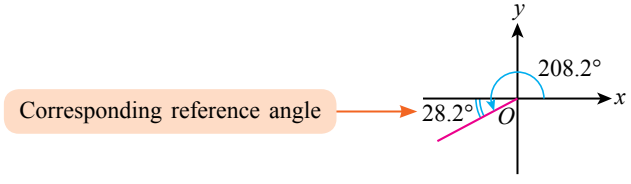
(a) $\sin 104° = +\sin (180° - 104°)$
$= +\sin 76°$
$= 0.9703$

sine values are positive in quadrant II



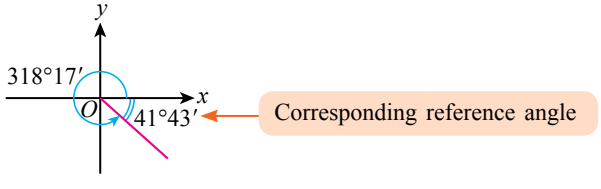
(b) $\cos 208.2° = -\cos (208.2° - 180°)$
$= -\cos 28.2°$
$= -0.8813$

cosine values are negative in quadrant III



(c) $\tan 318°17′ = -\tan (360° - 318°17′)$
$= -\tan 41°43′$
$= -0.8915$

tangent values are negative in quadrant IV



Smart TiPS

Quadrant I: **A**ll positive
Quadrant II: **s**in θ positive
Quadrant III: **t**an θ positive
Quadrant IV: **c**os θ positive

Tips to remember:
Use appropriate acronym.

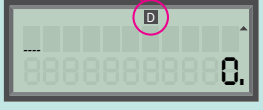
i-Technology

To determine the value of tan 41°43′ from the scientific calculator, press

tan 4 1 °′″
4 3 °′″ =

Make sure your scientific calculator is in "Deg" mode.

"Deg" mode



Self Practice 6.1b

1. Determine the value of each of the following based on the corresponding reference angle.

(a) sin 128°	(b) sin 236°	(c) sin 337°
(d) cos 196°	(e) cos 289°	(f) cos 127°
(g) tan 221°	(h) tan 134°	(i) tan 316°
(j) tan 321.4°	(k) sin 341.7°	(l) cos 99.3°
(m) cos 307°39′	(n) tan 102°38′	(o) sin 197°42′

How to determine the value of sine, cosine and tangent for angles in quadrants II, III and IV corresponding to angles 30°, 45° and 60°?

Example 5

Without using a scientific calculator, determine the value for each of the following based on the corresponding reference angle.

(a) cos 150° (b) tan 225° (c) sin 300°

Solution:

(a) $\cos 150° = -\cos(180° - 150°)$ — cosine values are negative in quadrant II

$= -\cos 30°$

$= -\frac{\sqrt{3}}{2}$

(b) $\tan 225° = +\tan(225° - 180°)$ — tangent values are positive in quadrant III

$= +\tan 45°$

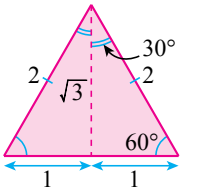
$= 1$

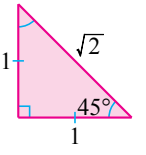
(c) $\sin 300° = -\sin(360° - 300°)$ — sine values are negative in quadrant IV

$= -\sin 60°$

$= -\frac{\sqrt{3}}{2}$

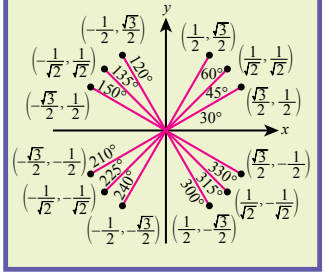
MEMORY BOX





Ratio \ Angle	30°	60°	45°
sin θ	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
cos θ	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tan θ	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1

Info Bulletin



Self Practice 6.1c

1. Determine the values of sine, cosine and tangent for each of the following angles without using a scientific calculator.

(a) 120°	(b) 135°	(c) 210°
(d) 240°	(e) 315°	(f) 330°

2. Given that $\sin\theta = \frac{\sqrt{3}}{2}$, calculate $\cos\theta$ and $\tan\theta$ without using a scientific calculator.

How do you determine the angle when the value of sine, cosine and tangent are given?

Learning Standard

Determine the angle when the value of sine, cosine and tangent are given.

Identify the quadrant based on the positive or negative sign on the value of trigonometric function. → Determine the corresponding reference angle. → Calculate angle θ based on the quadrant identified.

Example 6

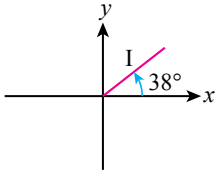
(a) Given that $\sin\theta = 0.6157$ and $0° \leqslant \theta \leqslant 360°$, calculate angle θ.
(b) Given that $\cos\theta = -0.4226$ and $0° \leqslant \theta \leqslant 360°$, calculate angle θ.
(c) Given that $\tan\theta = -1.4826$ and $0° \leqslant \theta \leqslant 360°$, calculate angle θ.

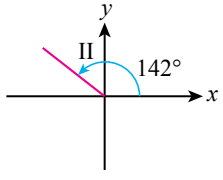
Solution:

(a) $\sin\theta = 0.6157$ ← Positive sign. θ is in quadrant I or II

Corresponding reference angle
$= \sin^{-1} 0.6157$
$= 38°$

$\theta = 38°$ or $(180° - 38°)$
$= 38°$ or $142°$

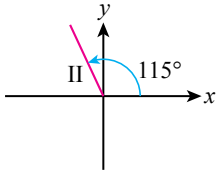


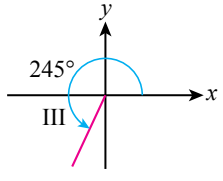


(b) $\cos\theta = -0.4226$ ← Negative sign. θ is in quadrant II or III

Corresponding reference angle
$= \cos^{-1} 0.4226$
$= 65°$

$\theta = (180° - 65°)$ or $(180° + 65°)$
$= 115°$ or $245°$

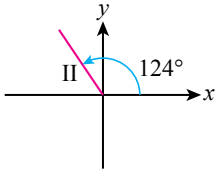


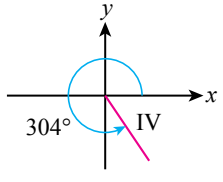


(c) $\tan\theta = -1.4826$ ← Negative sign. θ is in quadrant II or IV

Corresponding reference angle
$= \tan^{-1} 1.4826$
$= 56°$

$\theta = (180° - 56°)$ or $(360° - 56°)$
$= 124°$ or $304°$



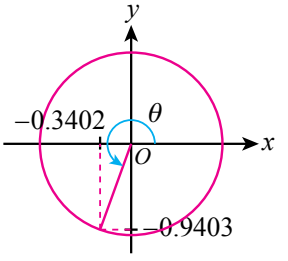


Self Practice 6.1d

1. Given that $0° \leqslant \theta \leqslant 360°$, calculate angle θ for each of the following. Round off the answer to 1 decimal place.

(a) $\sin \theta = 0.9397$ (b) $\cos \theta = 0.9336$ (c) $\tan \theta = 0.8391$
(d) $\tan \theta = -1.198$ (e) $\cos \theta = -0.6018$ (f) $\sin \theta = -0.7314$
(g) $\cos \theta = -0.5829$ (h) $\sin \theta = -0.8395$ (i) $\tan \theta = 0.7391$

2. The diagram on the right shows a unit circle with $0° \leqslant \theta \leqslant 360°$. Calculate angle θ. Round off the answer to 1 decimal place.



How to solve problems involving sine, cosine and tangent?

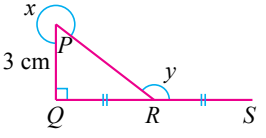
Learning Standard

Solve problems involving sine, cosine and tangent.

Example 7

In the diagram on the right, QRS is a straight line. Given that $QS = 8$ cm and $QR = RS$, calculate

(a) $\cos x$
(b) $\tan y$



Solution:

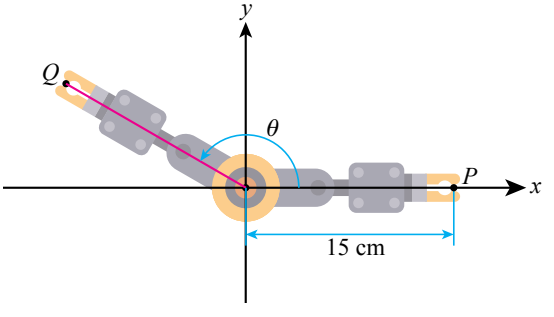
$QR = 8 \text{ cm} \div 2$ and $PR = \sqrt{3^2 + 4^2}$
$= 4$ cm $= 5$ cm

(a) $270° < x < 360°$,
hence, angle x is in quadrant IV.
$\cos x = +\cos \angle QPR$ ← $\angle QPR$ is the corresponding reference angle of angle x
$= +\frac{3}{5}$

(b) $90° < y < 180°$,
hence, angle y is in quadrant II.
$\tan y = -\tan \angle PRQ$ ← $\angle PRQ$ is the corresponding reference angle of angle y
$= -\frac{3}{4}$

Example 8

A group of students produces a robotic arm with the length of 15 cm as shown in the diagram on the right. The robotic arm is programmed to move an object from point P to point Q. Given that $\cos \theta = -0.866$ and $0° \leqslant \theta \leqslant 180°$, calculate the angle θ and the distance, in cm, between point P and point Q.



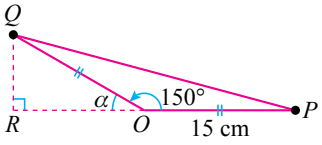
Solution:

$\cos \theta = -0.866$

Negative sign. θ is in quadrant II ($0° \leqslant \theta \leqslant 180°$)

Corresponding reference angle
$= \cos^{-1} 0.866$
$= 30°$

$\theta = 180° - 30°$
$= 150°$



$\alpha = 180° - 150°$
$= 30°$

$\sin 30° = \dfrac{QR}{15}$

$QR = 15 \sin 30°$
$= 7.5$ cm

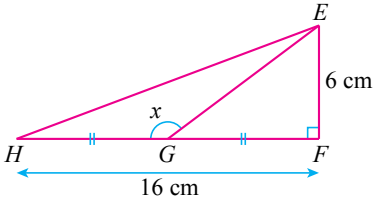
$\cos 30° = \dfrac{OR}{15}$

$OR = 15 \cos 30°$
$= 12.99$ cm

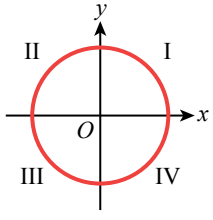
$PQ = \sqrt{QR^2 + PR^2}$
$= \sqrt{7.5^2 + 27.99^2}$
$= 28.98$ cm

Self Practice 6.1e

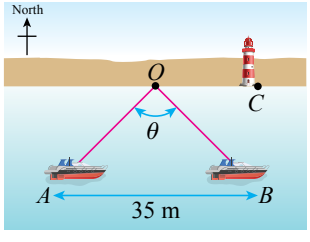
1. In the diagram on the right, HGF is a straight line. Given that $HG = GF$, calculate
(a) $\cos x$,
(b) $\sin x$.



2. Yong Ying plays a game of darts. The dartboard is round in shape and has four quadrants as shown in the diagram on the right. She finds that the darts she throws will usually hit the quadrants with the value of $\sin \theta = 0.809$. State
(a) the quadrants,
(b) the angles θ.



3. A ship sails from position A to position B such that $OA = OB$. The route of the ship is parallel to the sea bank. From position B, the ship will sail north towards the lighthouse, C. If $OA = 20$ m, calculate
(a) the angle θ,
(b) the distance, in m, between the ship in position B and the lighthouse,
(c) the value of $\tan \angle AOC$.



4. The diagram on the right shows an alarm clock.
(a) Calculate the angle θ moved through by the minute hand if the minute hand is being rotated anticlockwise from number 3 to number 10.
(b) Hence, calculate the value of $\cos \theta$.

6.2 The Graphs of Sine, Cosine and Tangent Functions

After doing a heart check-up at the hospital, the doctor will explain the condition of our heart based on the result of the electrocardiogram. Electrocardiogram graphs have the characteristics of trigonometric function graphs. What are the characteristics?

What are the characteristics of the graphs of trigonometric functions, $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $0° \leqslant x \leqslant 360°$?

Learning Standard

Draw graphs of trigonometric functions, $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $0° \leqslant x \leqslant 360°$, hence compare and contrast the characteristics of the graphs.

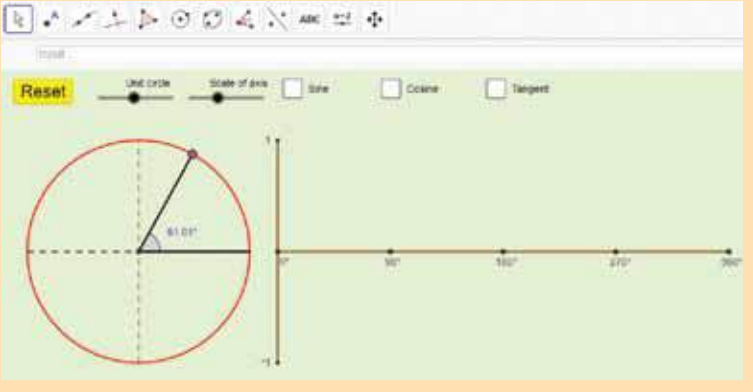
MIND MOBILISATION 3 Pairs

Aim: To draw graphs of trigonometric functions, $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $0° \leqslant x \leqslant 360°$, hence compare and contrast the characteristics of the graphs.

Steps:

Activity I

1. Open the file GGB602 for this activity.



Scan the QR code or visit bit.do/GGB602BI to obtain the GeoGebra file for this activity.

2. Click 'Sine' and then drag the red point.
3. Observe the shape, the value on y-axis when the angle is 0°, maximum value, minimum value and x-intercept of the graph formed.
4. Repeat Steps **2** to **3** with 'Cosine' and 'Tangent'.
   [Click 'Reset' if you want to erase the graph formed.]

Activity II

1. Open the worksheet by scanning the QR code for this activity. Complete the table given.

x	0°	15°	30°	45°	60°	75°	90°
sin x							
cos x							
tan x							

Scan the QR code or visit bit.do/WSChap6 to obtain the worksheet for this activity.

2. By using the scale of 2 cm to 30° on the x-axis and 2 cm to 1 unit on the y-axis, draw the graphs of the functions $y = \sin x$, $y = \cos x$ and $y = \tan x$ separately on graph papers.
3. Observe the shape, maximum value, minimum value, x-intercept and y-intercept of each of the graphs.
4. Complete the following table.

	Maximum value	Minimum value	x-intercept	y-intercept
$y = \sin x$				
$y = \cos x$				
$y = \tan x$				

Discussion:

For the graphs of sine, cosine and tangent,

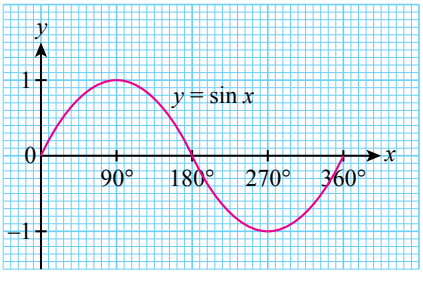
(a) what is the shape of each graph?

(b) what are the maximum value and minimum value of each graph?

(c) what are the x-intercept and y-intercept of each graph?

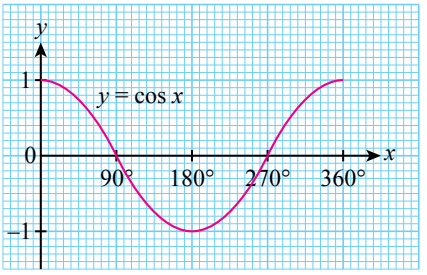
The results of Mind Mobilisation 3 show that the characteristics of the graphs of sine, cosine and tangent are as follows.

Sine graph, $y = \sin x$



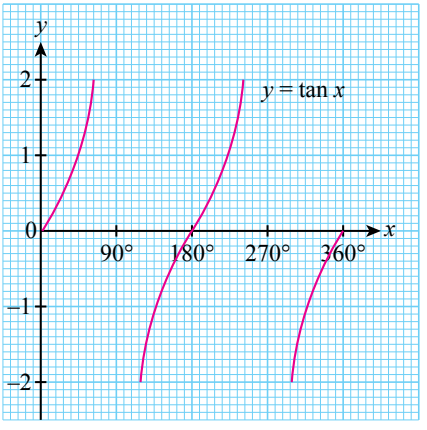
- has a maximum value of 1 when $x = 90°$ and a minimum value of –1 when $x = 270°$
- intercepts x-axis at $x = 0°$, 180° and 360° (x-intercept)
- intercepts y-axis at $y = 0$ (y-intercept)

Cosine graph, $y = \cos x$



- has a maximum value of 1 when $x = 0°$ and $360°$ and a minimum value of -1 when $x = 180°$
- intercepts x-axis at $x = 90°$ and $270°$ (x-intercept)
- intercepts y-axis at $y = 1$ (y-intercept)

Tangent graph, $y = \tan x$



- maximum value is ∞ and minimum value is $-\infty$
- intercepts x-axis at $x = 0°$, $180°$ and $360°$ (x-intercept)
- intercepts y-axis at $y = 0$ (y-intercept)
- the values of tan 90° and tan 270° are undefined

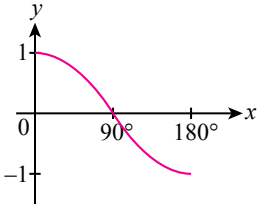
Critical Mind

- What is meant by an undefined value?
- Why are tan 90° and tan 270° undefined?

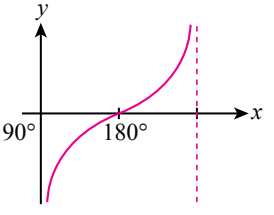
Self Practice 6.2a

1. Determine whether each of the following trigonometric graphs is a sine, cosine or tangent graph.

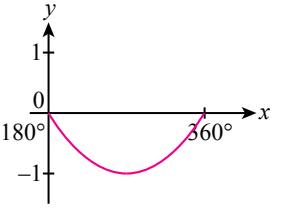
(a)



(b)

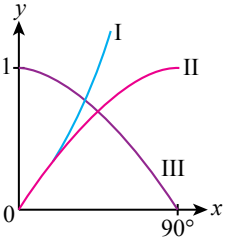


(c)



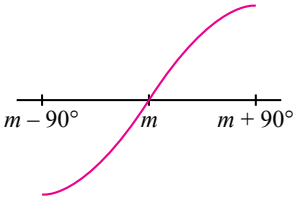
2. (a) Sketch the graph for $y = \sin x$ where $90° \leqslant x \leqslant 270°$.
(b) Sketch the graph for $y = \cos x$ where $45° \leqslant x \leqslant 225°$.

3. The diagram on the right shows three trigonometric graphs for angles between 0° and 90°.
(a) Identify the equations for graphs I, II and III.
(b) State the maximum value and minimum value of graphs I, II and III, if any.

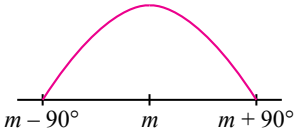


4. For each of the following trigonometric functions, state the value of x when the value of y is maximum and state the maximum value.
(a) $y = \sin x,\ 0° \leqslant x \leqslant 360°$
(b) $y = \cos x,\ 0° \leqslant x \leqslant 360°$
(c) $y = \tan x,\ 0° \leqslant x \leqslant 360°$

5. Each diagram below shows part of a graph of trigonometric function such that $0° \leqslant x \leqslant 360°$. State the function and the value of m.
(a)



(b)

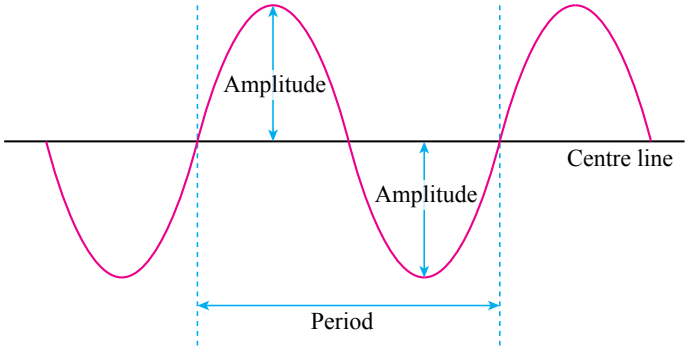


What are the effects of changes in constants a, b and c on the graphs of trigonometric functions $y = a \sin bx + c$, $y = a \cos bx + c$ and $y = a \tan bx + c$?

Learning Standard

Investigate and make generalisations about the effects of changes in constants a, b and c on the graphs of trigonometric functions:
(i) $y = a \sin bx + c$
(ii) $y = a \cos bx + c$
(iii) $y = a \tan bx + c$
for $a > 0$, $b > 0$.

The diagram below shows the characteristics of a periodic function, namely period and amplitude.



Smart TiPS

Amplitude can also be determined by the formula:

$$\left(\frac{\text{maximum value} - \text{minimum value}}{2}\right)$$

Period is the interval of one complete cycle. The trigonometric function is a periodic function. The graph of a trigonometric function repeats itself on a specific interval. For example, the graph of sine function repeats itself on an interval of 360°. We can say that the function $y = \sin x$ is a periodic function with a period of 360°. Amplitude is the maximum distance measured from the centre line.

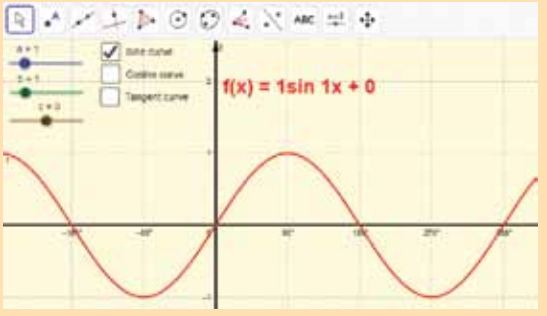
If given that the trigonometric functions $y = a \sin bx + c$, $y = a \cos bx + c$ and $y = a \tan bx + c$, what would happen to the shape and position of the graphs of the trigonometric functions if the values of constants a, b and c change?

MIND MOBILISATION 4 Group

Aim: To investigate and make generalisations about the effects of changes in constants a, b and c on the graphs of trigonometric functions $y = a \sin bx + c$, $y = a \cos bx + c$ and $y = a \tan bx + c$.

Steps:

1. Open the file GGB603 for this activity.



Scan the QR code or visit bit.do/GGB603BI to obtain the GeoGebra file for this activity.

2. Drag the slider of the value of a and observe the change on the graph displayed.
3. Drag the slider of the value of b and observe the change on the graph displayed.
4. Drag the slider of the value of c and observe the change on the graph displayed.
5. Repeat steps **2** to **4** for 'Cosine curve' and 'Tangent curve'.

Discussion:

What are your conclusions about the effects of changes in constants a, b and c on the graphs of trigonometric functions $y = a \sin bx + c$, $y = a \cos bx + c$ and $y = a \tan bx + c$ for $a > 0$, $b > 0$?

The results of Mind Mobilisation 4 show that
(a) when a changes, the maximum and minimum values change,
(b) when b changes, the graph will be compressed or expanded,
(c) when c changes, the graph will move vertically up or down.
The value of a affects the amplitude of the function, the value of b affects the period of the function and the value of c affects the position of the graph of the function.

In general,

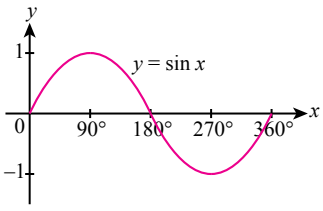
	$y = a \sin bx + c$, $a > 0$, $b > 0$	$y = a \cos bx + c$, $a > 0$, $b > 0$	$y = a \tan bx + c$, $a > 0$, $b > 0$
The value of a changes	• Maximum and minimum values change • Amplitude of function = a		• Graph curvature changes • No amplitude
The value of b changes	• The period of the function changes • When the value of b increases, the graph appears to compress horizontally, the period of the function decreases • The period of sine and cosine functions = $\frac{360°}{b}$; The period of tangent function = $\frac{180°}{b}$		
The value of c changes	• The position of the graph changes • When $c > 0$, the graph moves c units vertically up the x-axis • When $c < 0$, the graph moves c units vertically down the x-axis		

Example 9

The diagram on the right shows a graph of the function $y = \sin x$ for $0° \leqslant x \leqslant 360°$. Sketch each of the following trigonometric functions on the same axes.

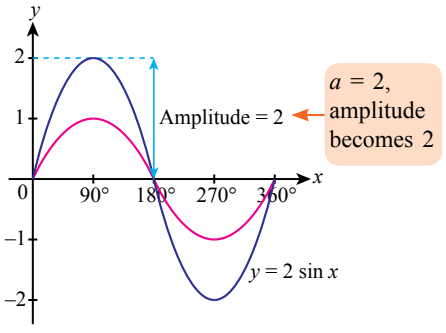
(a) $y = 2 \sin x$ (b) $y = \sin 2x$

(c) $y = \sin x + 1$ (d) $y = 2 \sin 2x$

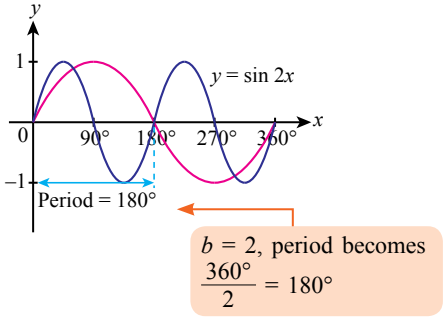


Solution:

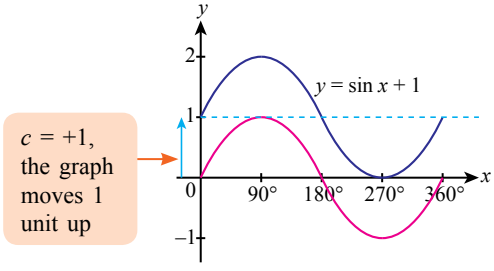
(a)



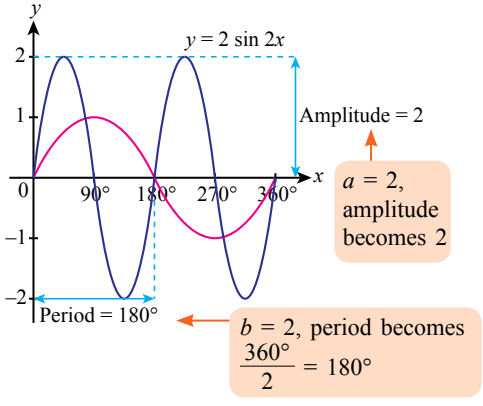
(b)



(c)



(d)



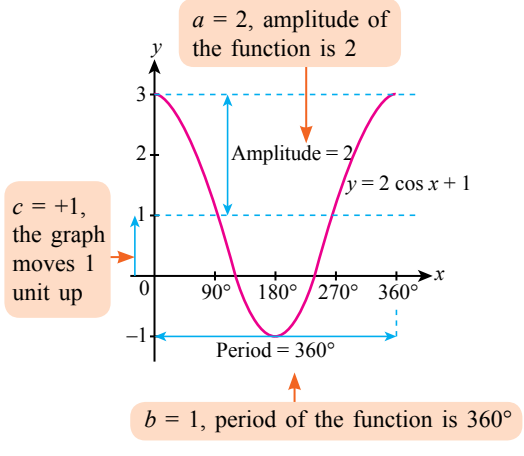
Example 10

Sketch each of the following trigonometric functions for $0° \leqslant x \leqslant 360°$.

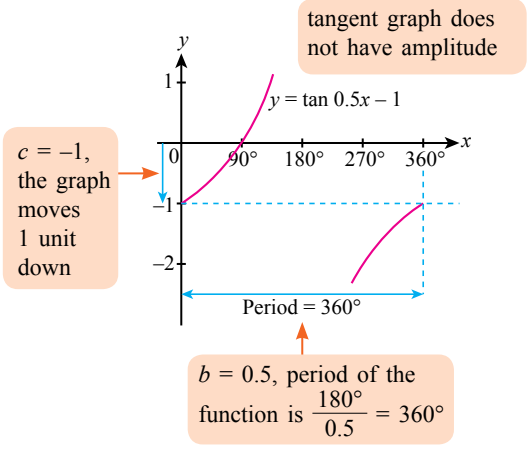
(a) $y = 2 \cos x + 1$ (b) $y = \tan 0.5x - 1$

Solution:

(a)



(b)



Self Practice 6.2b

1. Determine the amplitude and the period of each of the following trigonometric functions.

(a) $y = 4 \sin x$ (b) $y = 3 \sin 2x$ (c) $y = 2 \sin 3x - 4$

(d) $y = \cos 4x$ (e) $y = 4 \cos 2x$ (f) $y = 3 \cos 3x + 1$

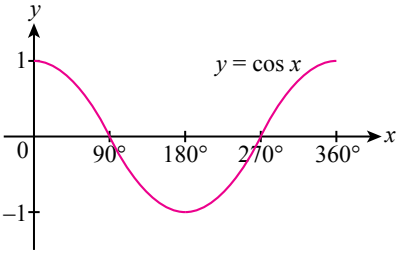
(g) $y = \frac{1}{3} \tan 3x$ (h) $y = 3 \tan \frac{1}{3}x$ (i) $y = 3 \tan 2x + 2$

2. The diagram on the right shows a graph of the function $y = \cos x$ for $0° \leqslant x \leqslant 360°$. Sketch each of the following trigonometric functions on the same axes.

(a) $y = \frac{1}{2} \cos x$

(b) $y = \cos \frac{x}{2}$

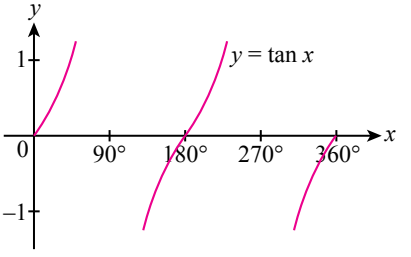
(c) $y = \cos x - 2$



3. The diagram on the right shows a graph of the function $y = \tan x$ for $0° \leqslant x \leqslant 360°$. Sketch each of the following trigonometric functions on the same axes.

(a) $y = \tan 2x$

(b) $y = \tan x + 2$

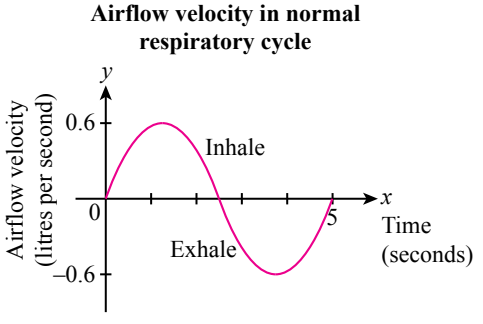


How to solve problems involving graphs of sine, cosine and tangent functions?

Learning Standard

Solve problems involving graphs of sine, cosine and tangent functions.

Example 11



The graph above shows a complete respiratory cycle. This cycle consists of the process of inhalation and exhalation. It occurs every 5 seconds. Airflow velocity is positive when we inhale and negative when we exhale. This velocity is measured in litre per second.

(a) If y represents the velocity of airflow after x seconds, state a function in the form of $y = a \sin bx + c$ which models the airflow in normal respiratory cycle shown as the graph above.

(b) What is the velocity of airflow, in litres per second, when the time is 7 seconds?

Solution:

(a) From the graph, amplitude = 0.6, therefore $a = 0.6$

period = 5 seconds, therefore $\frac{360°}{b} = 5$

$b = 72$

there is no movement of going up or down the x-axis, therefore $c = 0$

Hence, a function that models the airflow in normal respiratory cycle shown as the graph is $y = 0.6 \sin 72x$.

(b) $y = 0.6 \sin 72x$

When $x = 7$, $y = 0.6 \sin (72 \times 7)$

$y = 0.35$

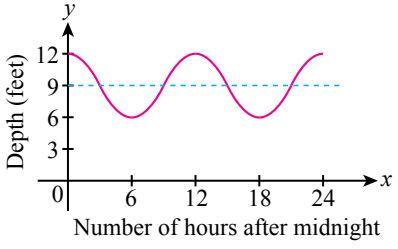
Hence, the airflow velocity is 0.35 litre per second when the time is 7 seconds.

Self Practice 6.2c

1. The graph on the right shows the depth of water recorded in a dockyard.

(a) If y represents the depth of water, in feet, and x represents the number of hours after midnight, use the function in the form of $y = a \cos bx + c$ to model the depth of water shown as the graph.

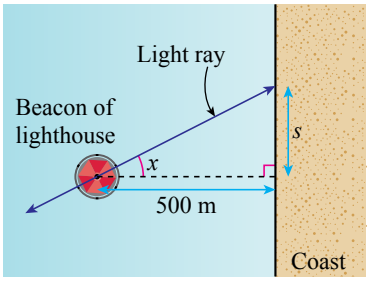
(b) At what time will the water be the deepest?



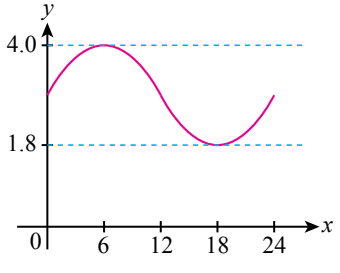
2. The diagram on the right shows the view from the top of a lighthouse and the coast. The beacon of the lighthouse sends out a light ray as shown in the diagram. When the beacon rotates, the light ray moves along the coast at an angle, x.

(a) Write a function for the distance s.

(b) State the amplitude and the period of the function.



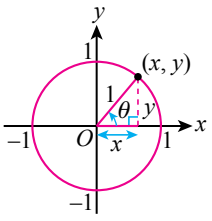
3. The graph on the right depicts the water level recorded at a port. It is given that y represents the water level, in m, and x represents time, in hours. State the trigonometric function of the graph in the form of $y = a \sin bx + c$.



Summary Arena

Sine, Cosine and Tangent

Unit circle

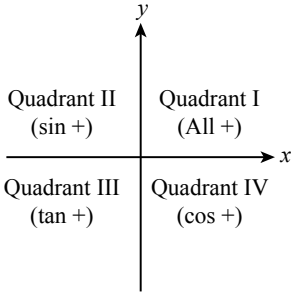


$\sin \theta$ = y-coordinate

$\cos \theta$ = x-coordinate

$\tan \theta = \dfrac{y\text{-coordinate}}{x\text{-coordinate}}$

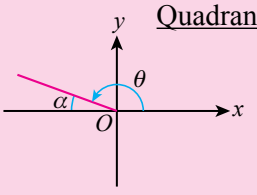
Signs of values of $\sin \theta$, $\cos \theta$ and $\tan \theta$



Corresponding reference angles

Let α be a corresponding reference angle

Quadrant II



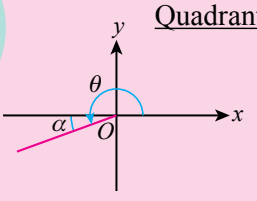
$\alpha = 180° - \theta$

$\sin \theta = +\sin \alpha$

$\cos \theta = -\cos \alpha$

$\tan \theta = -\tan \alpha$

Quadrant III



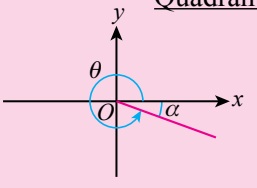
$\alpha = \theta - 180°$

$\sin \theta = -\sin \alpha$

$\cos \theta = -\cos \alpha$

$\tan \theta = +\tan \alpha$

Quadrant IV



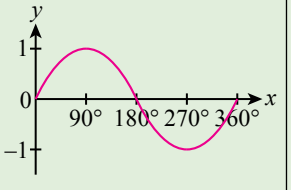
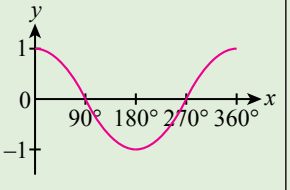
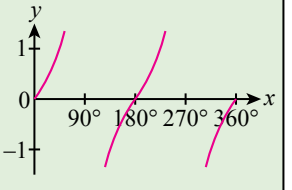
$\alpha = 360° - \theta$

$\sin \theta = -\sin \alpha$

$\cos \theta = +\cos \alpha$

$\tan \theta = -\tan \alpha$

Graphs of $\sin x$, $\cos x$ and $\tan x$ for $0° \leqslant x \leqslant 360°$

	$y = \sin x$	$y = \cos x$	$y = \tan x$
Shape of graph			
Maximum value	1	1	∞
Minimum value	–1	–1	$-\infty$
x-intercept	0°, 180°, 360°	90°, 270°	0°, 180°, 360°
y-intercept	0	1	0

Reflection

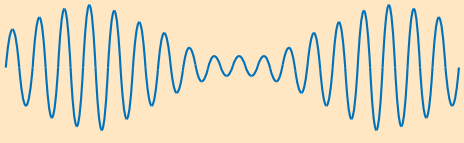
At the end of this chapter, I can

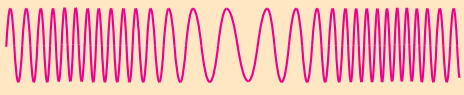
		
make and verify conjecture about the value of sine, cosine and tangent for angles in quadrants II, III and IV with the corresponding reference angle.		
determine the value of sine, cosine and tangent for angles in quadrants II, III and IV based on the corresponding reference angle.		
determine the angle when the value of sine, cosine and tangent are given.		
solve problems involving sine, cosine and tangent.		
draw graphs of trigonometric functions $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $0° \leqslant x \leqslant 360°$, hence compare and contrast the characteristics of the graphs.		
investigate and make generalisations about the effects of changes in constants a, b and c on the graphs of trigonometric functions: (i) $y = a \sin bx + c$ (ii) $y = a \cos bx + c$ (iii) $y = a \tan bx + c$ for $a > 0$, $b > 0$.		
solve problems involving graphs of sine, cosine and tangent functions.		

MINI PROJECT

Radio broadcasting is an example of electronic communication that can be found today. Sounds like music and voices that we hear from the radio are broadcasted in the form of waves. In the case of AM radio, sound is transmitted through amplitude modulation whereas for FM radio, sound is transmitted through frequency modulation.

In groups of four, do a brief report that explains the difference between amplitude modulation and frequency modulation based on the characteristics of the graphs of the trigonometric functions that you have learnt. You are encouraged to use appropriate graphs, tables, and mind maps to present your group report.





Extensive Practice

Scan the QR code or visit bit.do/QuizE06 for interactive quiz

UNDERSTAND

1. State the relationship between each of the following trigonometric functions with its corresponding reference angle.
(a) tan 154° (b) sin 234° (c) cos 314°

2. Determine the value of each of the following based on its corresponding reference angle.
(a) cos 116° (b) tan 211°38′ (c) sin 305.6°

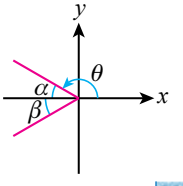
3. Given that $\tan \theta = -0.7265$ and $0° \leqslant \theta \leqslant 360°$, calculate the angle θ.

4. Sketch the graph of $y = \cos x$ for $90° \leqslant x \leqslant 270°$.

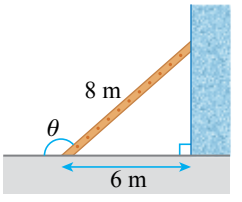
MASTERY

5. State the maximum and minimum values of the graph of function $y = 3 \sin 2x - 1$ for $0° \leqslant x \leqslant 360°$.

6. In the diagram on the right, $\theta = 150°$ and $\alpha = \beta$. Determine the values of
(a) $\cos \alpha$
(b) $\tan \beta$

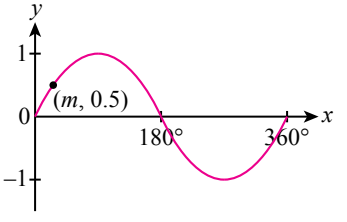


7. The diagram on the right shows a piece of wood with a length of 8 m leaning against a vertical wall. The horizontal distance from the wall to the wood is 6 m. Calculate the value of $\sin \theta$.

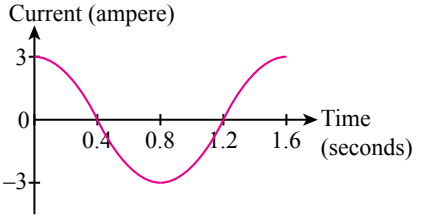


8. Sketch the graph of the function $y = 3 \sin 2x + 1$ for $0° \leqslant x \leqslant 360°$.

9. The diagram on the right shows a graph of a trigonometric function for $0° \leqslant x \leqslant 360°$.
(a) Write the trigonometric function.
(b) State the value of m.



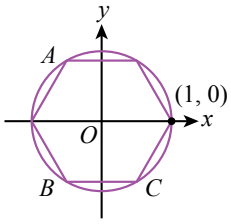
10. The diagram on the right shows a graph obtained on the screen of an oscilloscope when a supply of alternating current is connected to it.
(a) Which type of trigonometric functions is represented by the graph?
(b) State the amplitude of the current.
(c) State the period of the current.



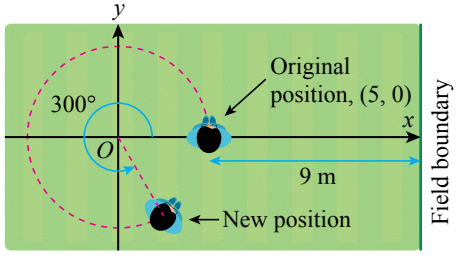
CHAPTER 6

CHALLENGE

11. A regular hexagon is drawn in a unit circle as shown in the diagram on the right. If one of the vertices of the hexagon is at (1, 0), determine the coordinates of vertices A, B and C.



12. Your school band performs on the field. Band members make a circular formation with a diameter of 10 m. Assume that you are one of the members and your original position is 9 m from the field boundary. After you have moved, your new position is as shown in the diagram on the right. If you are required to move to the field boundary from your new position, what is the shortest distance, in m, would you walk through?

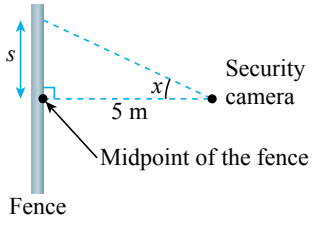


13. The table below shows the height of Ming Seng's position from the horizontal ground as he rides on the Ferris wheel.

Time (minutes)	0	2	4	6	8
Height (m)	20	31	20	9	20

(a) Based on the table, state the type of trigonometric function that it can represent.
(b) Hence, if y is the height of Ming Seng's position from the horizontal ground, in m, and x is the time in minutes, sketch the graph and state the trigonometric function representing the information above.

14. The diagram on the right shows a security camera in front of a fence of an apartment. The camera is mounted on a pole located 5 m from the midpoint of the fence. Write a trigonometric function that expresses the distance, s, in m, along the fence from its midpoint in terms of x.



EXPLORING MATHEMATICS

Do you know that trigonometric functions can be applied when programming a character which is jumping in a computer game?

Try to relate the movement patterns of computer game characters with trigonometric functions. Get the relevant information from legitimate sources such as interview a programmer, refer to a reading material or others. Write a short journal on

- the characteristics of trigonometric functions,
- the use of trigonometric functions in computer games,
- two other examples of real-life applications involving trigonometric functions.

CHAPTER 7

Measures of Dispersion for Grouped Data

What will you learn?

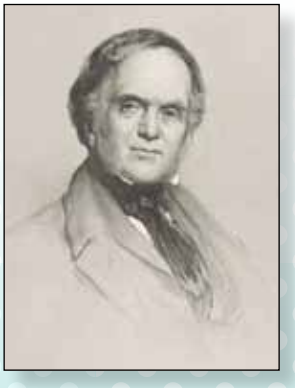
- Dispersion
- Measures of Dispersion

Why study this chapter?

Statistical analysis such as measure of dispersion is widely applied in various fields, including medicine, agriculture, finance, social science and many more. The career fields that apply statistical analysis include biometrics, actuarial science and financial analysis that use big data to obtain statistical values, and hence represent the data in statistical graphs.

Do you know?

William Playfair (1759-1823) was a Scottish economist who used various common statistical graphs in his book, The Commercial and Political Atlas, published in 1786.

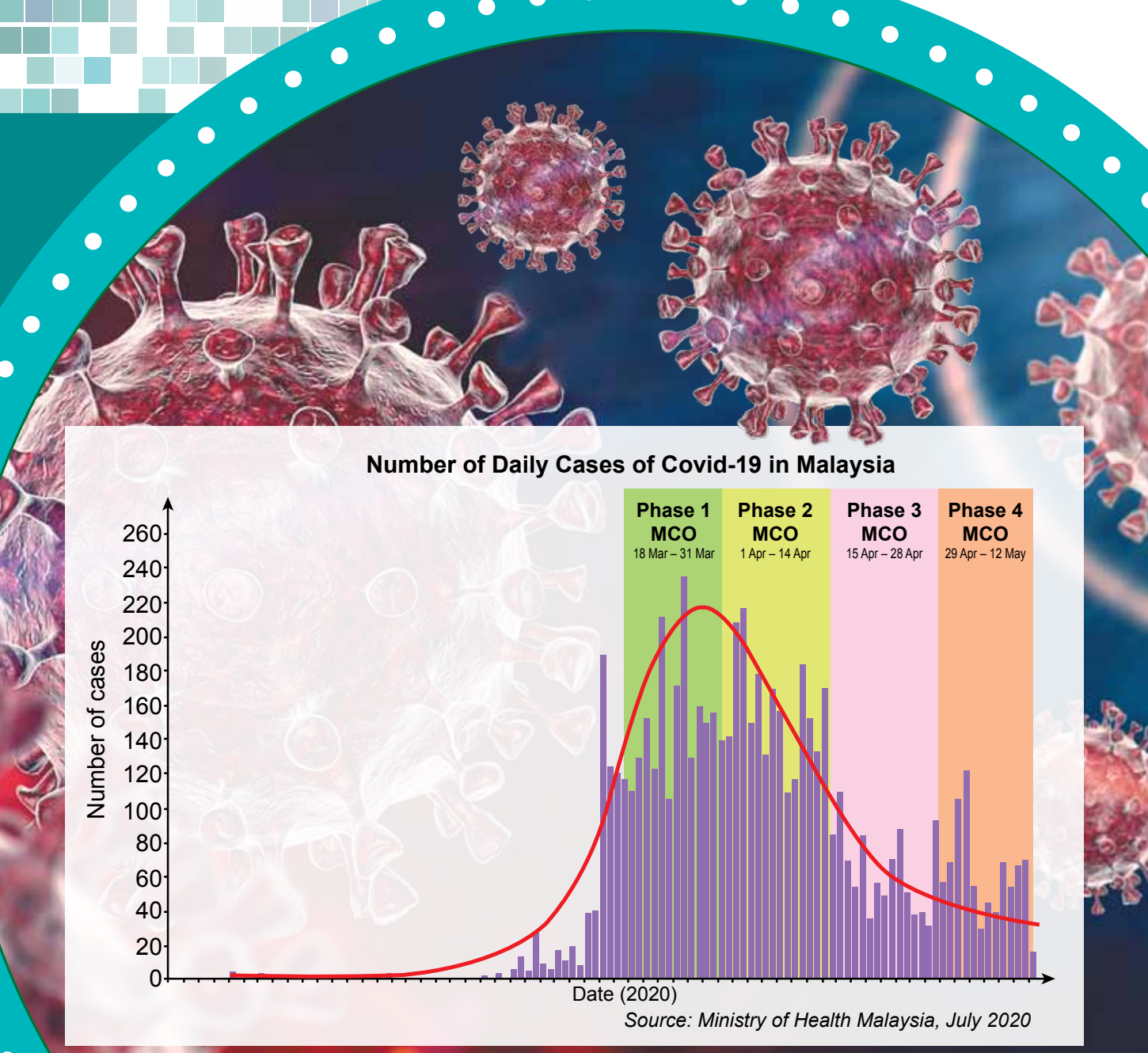


For more information:

bit.do/DoYouKnowChap7

WORD BANK

grouped data	*data terkumpul*
histogram	*histogram*
cumulative histogram	*histogram longgokan*
cumulative frequency	*kekerapan longgokan*
quartile	*kuartil*
ogive	*ogif*
statistical investigation	*penyiasatan statistik*
percentile	*persentil*
frequency polygon	*poligon kekerapan*



Number of Daily Cases of Covid-19 in Malaysia

Source: Ministry of Health Malaysia, July 2020

The outbreak of the Covid-19 pandemic in early 2020 has forced Malaysian to adjust to a new normal. The swift and efficient action taken by the authorities in tackling the pandemic has helped Malaysia to control the increasing number of patients infected by the virus. Malaysia is successful in flattening the curve of the number of daily infected cases by issuing the Movement Control Order (MCO). In your opinion, how will the shape of the graph be if MCO has not been implemented?

7.1 Dispersion

How to construct histogram and frequency polygon?

Learning Standard

Construct histogram and frequency polygon for a set of grouped data

In Form 4, you have learnt about the ways to interpret the dispersion of ungrouped data based on the stem-and-leaf plots and dot plots. We can observe the dispersion for a grouped data by constructing histogram and frequency polygon. Prior to that, you need to know the class interval, lower limit, upper limit, midpoint, lower boundary, upper boundary and cumulative frequency that can be obtained from a frequency table.

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Class interval is the range of a division of data.

MIND MOBILISATION 1 Group

Aim: To recognise the lower limit, upper limit, midpoint, lower boundary and upper boundary of a set of data.

Steps:

The data shows the amount of daily pocket money in RM, received by 20 pupils on a particular day.

8	10	4	7	1
5	2	8	11	4
5	7	15	3	4
14	12	7	11	9

1. Identify the smallest data and the largest data.
2. By referring to the data, group the data into 3, 4, 5 or 6 parts in sequence. For example, a group of three uniform parts means 1 – 5, 6 – 10 and 11 – 15.
3. By using the tally method, choose and insert the data according to the parts of the group.
4. Based on each part of the data, determine
   (a) the lower limit (the smallest value in a part of the data) and the upper limit (the largest value in a part of the data),
   (b) the midpoint of each part of the data,
   (c) (i) the middle value between the lower limit of a part and the upper limit of the part before it,
   (ii) the middle value between the upper limit of a part and the lower limit of the part after it.
5. Complete the frequency table with the results of steps 3, 4(a), 4(b), 4(c)(i) and 4(c)(ii) as shown below.

Pocket money (RM)	Frequency	Step 4(a)		Step 4(b)	Step 4(c)	
		Lower limit	Upper limit	Midpoint	(i)	(ii)

Discussion:

Discuss and write down the definition to determine the lower limit, upper limit, midpoint, lower boundary and upper boundary of a set of data.

The results of Mind Mobilisation 1 show that;

Size of class interval
$= \left(\frac{\text{Largest data value} - \text{Smallest data value}}{\text{Number of classes}}\right)$

Lower limit is the smallest value and upper limit is the largest value of a class.

$\text{Midpoint} = \left(\frac{\text{Lower limit} + \text{Upper limit}}{2}\right)$

Lower boundary
$= \left(\frac{\text{Upper limit of the class before it} + \text{Lower limit of the class}}{2}\right)$

Upper boundary
$= \left(\frac{\text{Upper limit of the class} + \text{Lower limit of the class after it}}{2}\right)$

Example 1

The data on the right shows the heights, to the nearest cm, of a group of Form 5 pupils.

153	168	163	157
158	161	165	162
145	150	158	156
166	163	152	155
158	173	148	164

(a) Determine the class intervals for the data, if the number of classes required is 6.

(b) Construct a frequency table based on the information in (a). Hence, complete the frequency table with the lower limit, upper limit, midpoint, lower boundary and upper boundary.

i-Technology

Scan the QR code or visit bit.do/WSChap7i to explore ways to organise raw data in frequency table by using spreadsheet.

Solution:

(a) The largest data is 173 and the smallest data is 145.
If the number of classes is 6, then the size of each class interval

$= \frac{173 - 145}{6}$
$= 4.7 \approx 5$

Size of class interval
$= \left(\frac{\text{Largest data value} - \text{Smallest data value}}{\text{Number of classes}}\right)$

Therefore, the class intervals are 145 – 149, 150 – 154, 155 – 159, 160 – 164, 165 – 169 and 170 – 174.

(b)

Height (cm)	Frequency	Lower limit	Upper limit	Midpoint	Lower boundary	Upper boundary
145 – 149	2	145	149	147	144.5	149.5
150 – 154	3	150	154	152	149.5	154.5
155 – 159	6	155	159	157	154.5	159.5
160 – 164	5	160	164	162	159.5	164.5
165 – 169	3	165	169	167	164.5	169.5
170 – 174	1	170	174	172	169.5	174.5

For a grouped data in uniform class intervals, the size of class interval can be calculated using two methods.

Method 1: The difference between the lower limits or the upper limits of two consecutive classes.

Size of class interval of the first two classes
= 150 – 145 ← Lower limit of class 145 – 149
= 5 ↑ Lower limit of class 150 – 154
or
= 154 – 149 ← Upper limit of class 145 – 149
= 5 ↑ Upper limit of class 150 – 154

Method 2: The difference between the upper boundary and the lower boundary of a class interval.

Size of class interval of the first class
= 149.5 – 144.5 ← Lower boundary of class 145 – 149
= 5 ↑ Upper boundary of class 145 – 149

Smart TIPS

To determine the size of class interval, avoid using lower and upper limits of a class. For example, for class interval 145 –149, the size of class interval
= 149 – 145
= 4 (Not true)

The cumulative frequency of a data can also be obtained from a frequency table. The **cumulative frequency** of a class interval is the sum of the frequency of the class and the total frequency of the classes before it. This gives an ascending cumulative frequency.

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In Example 1, class 150 – 154 is actually inclusive of the values from 149.5 to 154.5 because the data is a continuous data. The lower boundary 149.5 and the upper boundary 154.5 are used to separate the classes so that there are no gaps between 149 cm and 150 cm, also 154 cm and 155 cm.

Example 2

Construct a cumulative frequency table from the frequency table below.

Age	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59
Frequency	4	5	8	7	3

Solution:

Age	Frequency	Cumulative frequency
10 – 19	4	4
20 – 29	5	9
30 – 39	8	17
40 – 49	7	24
50 – 59	3	27

This value of 17 means there are 17 people aged 39 years old and below

Info Bulletin

- Continuous data is a data measured on a continuous scale. For example, the time taken by pupils to buy food at the canteen, and the pupils' heights.
- Discrete data is a data involving counting. For example, the number of pupils in Mathematics Club.

Histogram

Histogram is a graphical representation in which the data is grouped into ranges by using contiguous bars. The height of the bar in histogram represents the frequency of a class. Steps for constructing a histogram:

Find the lower boundary and upper boundary of each class interval. → Choose an appropriate scale on the vertical axis. Represent the frequencies on the vertical axis and the class boundaries on the horizontal axis. → Draw bars that represent each class where the width is equal to the size of the class and the height is proportionate to the frequency.

Frequency polygon

A frequency polygon is a graph that displays a grouped data by using straight lines that connect midpoints of the classes which lie at the upper end of each bar in a histogram. Steps for constructing a frequency polygon:

Mark the midpoints of each class on top of each bar. → Mark the midpoints before the first class and after the last class with zero frequency. → Draw straight lines by connecting the adjacent midpoints.

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Histogram and frequency polygon can only be constructed by using continuous data.

Example 3

The frequency table below shows the speed of cars in km h^{-1}, recorded by a speed trap camera along a highway in a certain duration. Represent the data with a histogram and frequency polygon by using a scale of 2 cm to 10 km h^{-1} on the horizontal axis and 2 cm to 10 cars on the vertical axis.

Speed (km h^{-1})	70 – 79	80 – 89	90 – 99	100 – 109	110 – 119	120 – 129
Number of cars	5	10	20	30	25	10

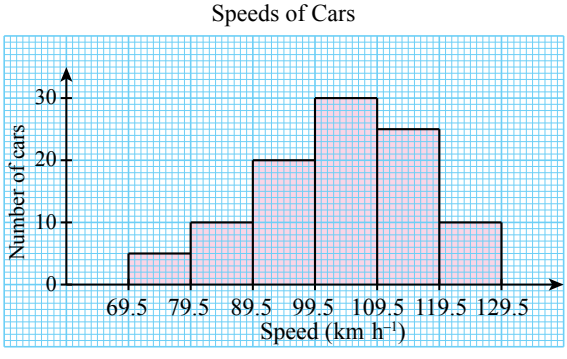
Solution:

Speed (km h^{-1})	Number of cars	Midpoint	Lower boundary	Upper boundary
70 – 79	5	74.5	69.5	79.5
80 – 89	10	84.5	79.5	89.5
90 – 99	20	94.5	89.5	99.5
100 – 109	30	104.5	99.5	109.5
110 – 119	25	114.5	109.5	119.5
120 – 129	10	124.5	119.5	129.5

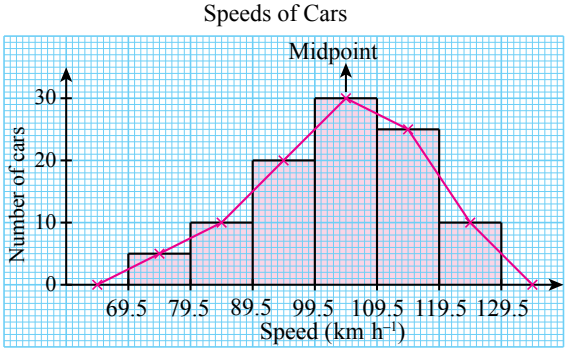
Critical Mind

By using the frequency polygon, explain the speed of cars of more than 90 km h^{-1}.

Histogram:



Frequency polygon:



The frequency polygon can also be constructed without constructing a histogram. Steps for constructing a frequency polygon from a frequency table:

Add one class interval before the first class and after the last class with zero frequency. → Find the midpoint of each class interval. → Choose an appropriate scale on the vertical axis. Represent the frequencies on the vertical axis and the midpoints on the horizontal axis. → Mark the midpoint with the corresponding frequency. → Connect each midpoint with a straight line.

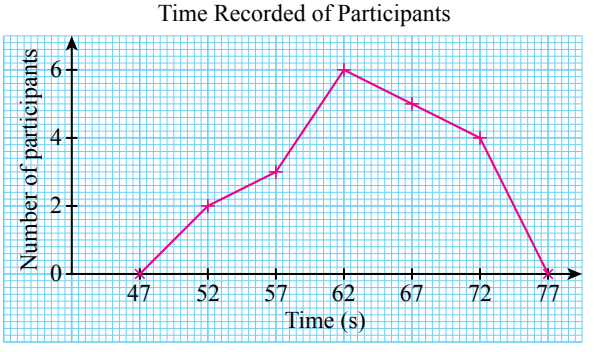
Example 4

The frequency table below shows the time in seconds, recorded by 20 participants in a qualifying round of a swimming competition. Represent the data with a frequency polygon by using a scale of 2 cm to 5 seconds on the horizontal axis and 2 cm to 2 participants on the vertical axis.

Time recorded (s)	50 – 54	55 – 59	60 – 64	65 – 69	70 – 74
Number of participants	2	3	6	5	4

Solution:

Time recorded (s)	Number of participants	Midpoint
45 – 49	0	47
50 – 54	2	52
55 – 59	3	57
60 – 64	6	62
65 – 69	5	67
70 – 74	4	72
75 – 79	0	77



Add a class interval with zero frequency before the first class and after the last class

Self Practice 7.1a

1. The data below shows the time taken by 50 pupils to go to school from their houses. The time recorded is in the nearest minute.

6	15	32	16	18	31	38	20	17	32
18	8	25	35	13	24	14	8	8	25
16	25	30	10	18	14	14	10	25	30
23	30	12	18	6	23	1	15	30	12
40	15	5	14	22	49	12	19	33	25

Construct a frequency table such that there are 5 classes. Then, state the lower limit, upper limit, midpoint, lower boundary and upper boundary of each class interval.

2. The frequency table below shows the masses in kg, of new-born babies in a hospital in a month. State the midpoint, lower limit, upper limit, lower boundary, upper boundary and cumulative frequency of the data.

Mass (kg)	2.0 – 2.4	2.5 – 2.9	3.0 – 3.4	3.5 – 3.9	4.0 – 4.4
Number of babies	9	15	24	20	10

3. The frequency table below shows the number of hours of sleep per day of a group of workers in a factory. By using a scale of 2 cm to 1 hour on the horizontal axis and 2 cm to 20 workers on the vertical axis, construct a histogram and frequency polygon on the same graph to represent the data.

Number of hours of sleep per day	4.05 – 5.04	5.05 – 6.04	6.05 – 7.04	7.05 – 8.04	8.05 – 9.04	9.05 – 10.04	10.05 – 11.04
Number of workers	2	4	22	64	90	14	2

4. The frequency table below shows the height in m, of sugar cane plants or also known as *Saccharum officinarum* taken from a plantation. Represent the data with a frequency polygon by using a scale of 2 cm to 1 m on the horizontal axis and 2 cm to 10 sugar cane plants on the vertical axis.

Height (m)	1.0 – 1.9	2.0 – 2.9	3.0 – 3.9	4.0 – 4.9	5.0 – 5.9	6.0 – 6.9
Number of sugar cane plants	25	33	46	50	44	36

How to compare and interpret the dispersions based on histogram and frequency polygon?

Learning Standard

Compare and interpret the dispersions of two or more sets of grouped data based on histogram and frequency polygon, hence make conclusion.

Distribution shapes of data

When describing a grouped data, it is important to be able to recognise the shapes of the distribution. The distribution shapes can be identified through a histogram or frequency polygon.

MIND MOBILISATION 2 Group

Aim: To explore the possible shapes of a distribution.

Steps:

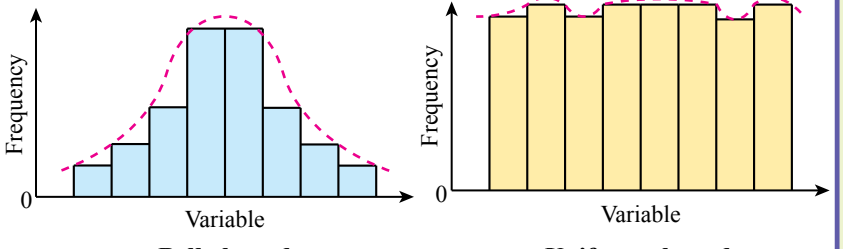
1. Divide the class into groups.
2. Open the worksheet by scanning the QR code. Each group is given the worksheet.
3. In the group, classify the distribution shapes into two categories, symmetrical or skewed.

Discussion:
Can you differentiate between symmetrical and skewed shapes?

Scan the QR code or visit bit.do/WSChap7ii to obtain the worksheet.

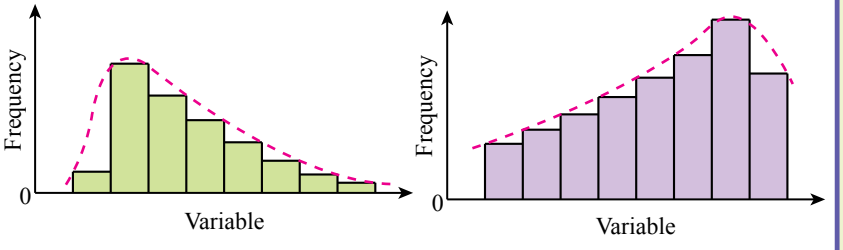
The results of Mind Mobilisation 2 show that a distribution is symmetric if the shape and size of the distribution are almost the same when divided into two parts, left and right. The shape of distribution is skewed if one tail of the histogram is longer than the other tail.

Symmetric Histogram



Bell-shaped **Uniform-shaped**

Skewed Histogram

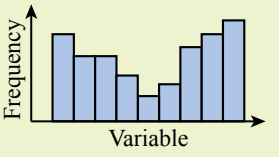


Right-skewed **Left-skewed**

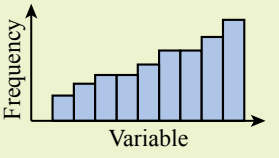
Info Bulletin

Other distribution shapes:

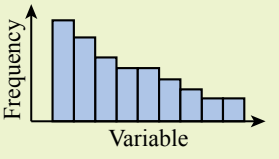
(i) U-shaped



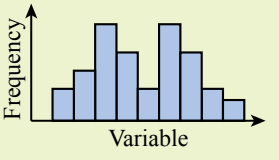
(ii) J-shaped



(iii) Reverse J-shaped



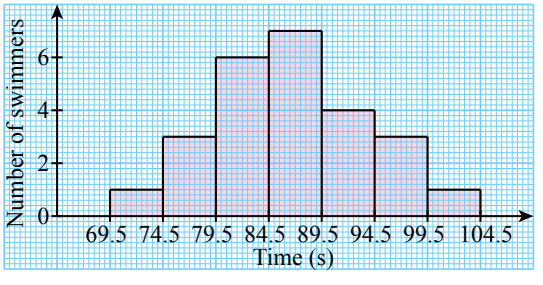
(iv) Bimodal



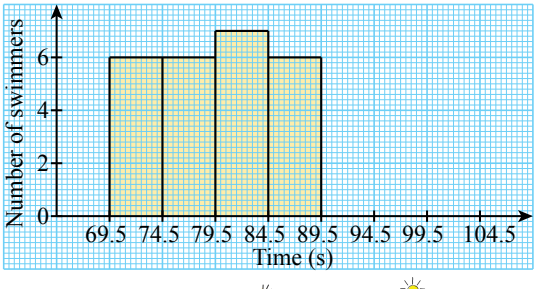
Example 5

The diagram below shows two histograms representing the time taken by 25 swimmers to complete two different events.

100 m Backstroke



100 m Freestyle



(a) State the distribution shape of the histogram for the two events.

(b) Which event has a wider dispersion of the time taken? Give your reason.

(c) Between backstroke and freestyle, in which event did the swimmers perform better?

Smart TiPS

Distributions are most often not perfectly shaped. Therefore, it is necessary to identify an overall pattern.

Solution:

(a) The histogram for the 100 m backstroke shows a bell-shaped distribution and for 100 m freestyle shows a uniform distribution.

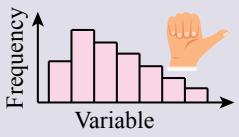
(b) The 100 m backstroke event has a wider dispersion because the difference of the time recorded is larger, that is 30 seconds (102 s – 72 s).

(c) 100 m freestyle. This is because most of the swimmers recorded a better time.

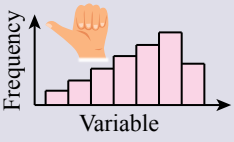
Smart TiPS

To determine distribution shape using hands:

(i) Skew to the right



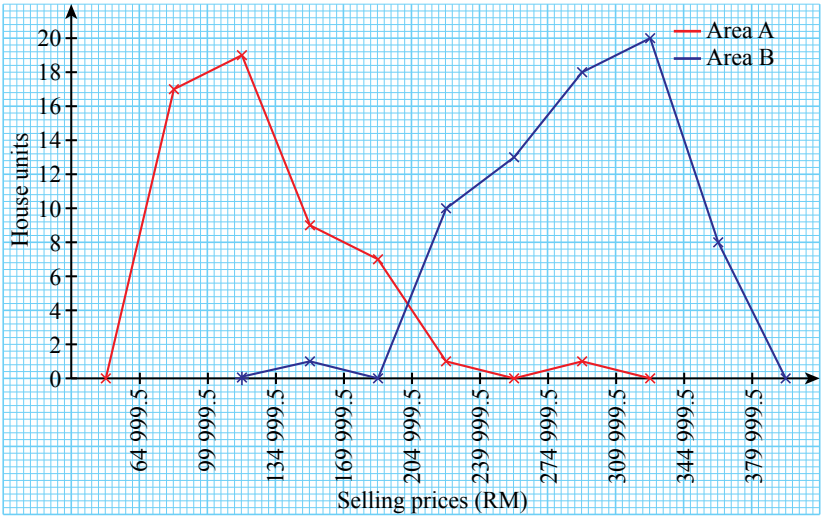
(ii) Skew to the left



Example 6

The frequency polygon below shows the selling prices of the houses that were sold in two different areas in the last six months.

Selling Prices of Houses in Area A and Area B



(a) State the distribution shapes in the two areas.

(b) Compare the dispersions of the house prices in the two areas.

(c) In your opinion, which area represents an urban area and which area represents a rural area?

Solution:

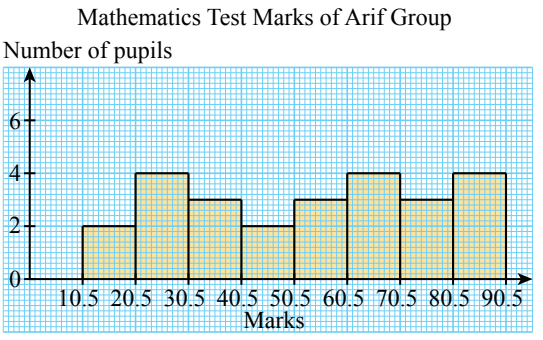
(a) The distribution shape of the selling prices in area A is skewed to the right whereas in area B is skewed to the left.

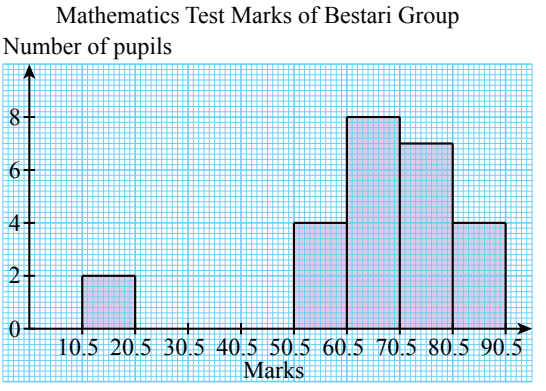
(b) The dispersions of the selling prices in area A and area B are approximately the same even though their distribution shapes are different.

(c) Area A represents a rural area because most of the selling prices are lower whereas Area B represents an urban area because most of the selling prices are higher.

Self Practice 7.1b

1. The diagram below shows two histograms of Mathematics test marks obtained by two groups, Arif and Bestari.



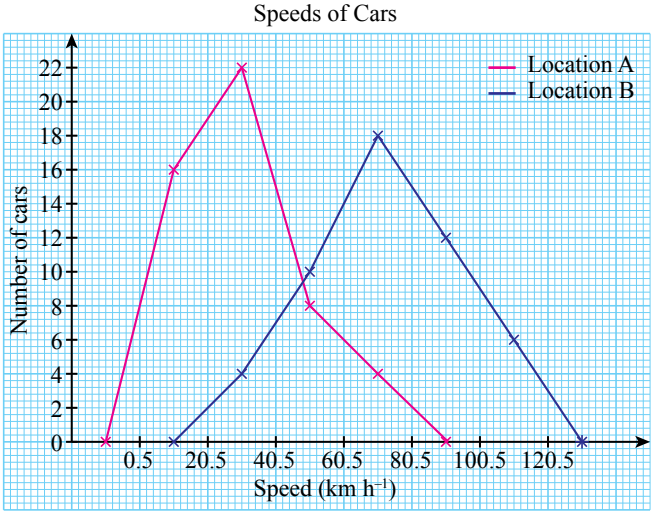


(a) State the distribution shape of the histogram for the two groups.

(b) Compare the dispersions of test marks between the two groups.

(c) Which group shows better results? Give your reason.

2. The diagram below shows the survey results of the traffic flow in two different locations. Each location records the speeds of 50 cars.



(a) State the distribution shapes in both locations.

(b) Compare the dispersions of the car speeds in both locations.

(c) In your opinion, which location is a highway and which location is a housing area?

How to construct an ogive for a set of grouped data?

Learning Standard

Construct an ogive for a set of grouped data and determine the quartiles.

Besides histogram and frequency polygon, a frequency distribution can also be displayed by drawing a cumulative frequency graph, also known as an ogive. When the cumulative frequencies of a data are plotted and connected, it will produce an S-shaped curve. Ogives are useful for determining the quartiles and the percentiles. We will learn how to use an ogive for this purpose in the next section.

Steps for constructing an ogive:

Add one class before the first class with zero frequency. Find the upper boundary and the cumulative frequency for each class. → Choose an appropriate scale on the vertical axis to represent the cumulative frequencies and the horizontal axis to represent the upper boundaries. → Plot the cumulative frequency with the corresponding upper boundary. → Draw a smooth curve passing through all the points.

Quartile

For a grouped data with number of data N, the quartiles can be determined from the ogive. Q_1, Q_2 and Q_3 are the values that correspond to the cumulative frequency $\frac{N}{4}$, $\frac{N}{2}$ and $\frac{3N}{4}$ respectively.

Example 7

The frequency table on the right shows the salt content of 60 types of food.

(a) Construct an ogive to represent the data.

(b) From your ogive, determine
  (i) the first quartile
  (ii) the median
  (iii) the third quartile

Salt content (mg)	Frequency
100 – 149	4
150 – 199	11
200 – 249	15
250 – 299	21
300 – 349	8
350 – 399	1

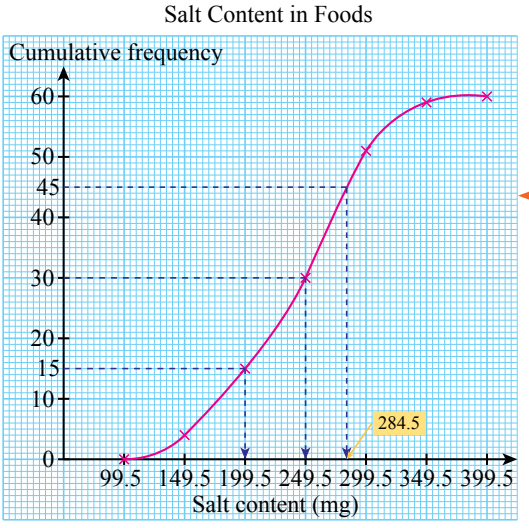
Solution:

(a)

Salt content (mg)	Frequency	Upper boundary	Cumulative frequency
50 – 99	0	99.5	0
100 – 149	4	149.5	4
150 – 199	11	199.5	15
200 – 249	15	249.5	30
250 – 299	21	299.5	51
300 – 349	8	349.5	59
350 – 399	1	399.5	60

MEMORY BOX

- Quartiles are values that divide a set of data into four equal parts. Each set of data has three quartiles, which are Q_1, Q_2 (median) and Q_3.
- The first quartile Q_1, also known as the lower quartile, is the middle value of the lower half of the data before the median or a quartile that contains 25% of the data.
- The second quartile, Q_2, also known as median is the middle value of a set of data.
- The third quartile, Q_3, also known as the upper quartile, is the middle value of the upper half of the data after the median or a quartile that contains 75% of the data.



Steps to determine the quartiles:

1. Number of data, $N = 60$, therefore $\frac{N}{4} = 15$, $\frac{N}{2} = 30$ and $\frac{3N}{4} = 45$.
2. Draw a horizontal line from the axis of cumulative frequency at 15 until it intersects the ogive.
3. From the intersection point in step 2, draw the vertical line down until it meets the axis of salt content at the horizontal axis.
4. The value of the salt content obtained is the value of Q_1.
5. Repeat steps 2 to 4 for the values of 30 and 45 to obtain the values of Q_2 and Q_3.

(b) $\frac{1}{4} \times 60 = 15$

From the graph, the first quartile, $Q_1 = 199.5$ mg

The salt content of 15 types of food are less or equal to 199.5 mg

$\frac{1}{2} \times 60 = 30$

From the graph, the median, $Q_2 = 249.5$ mg

The salt content of 30 types of food are less or equal to 249.5 mg

$\frac{3}{4} \times 60 = 45$

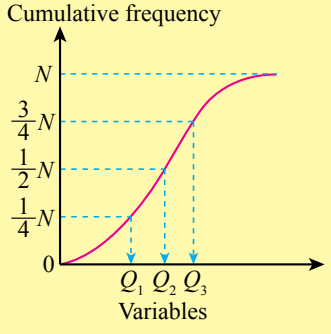
From the graph, the third quartile, $Q_3 = 284.5$ mg

The salt content of 45 types of food are less or equal to 284.5 mg

Info Bulletin

The average salt intake per day among Malaysians is 7.9 g (1.6 teaspoons). This is above the level recommended by the World Health Organization (WHO), which is less than 5 g (one teaspoon) per day.

From Example 7, the first quartile, median and third quartile of a grouped data can be determined by using an ogive.



The first quartile position, Q_1 = $\frac{1}{4}$ × total frequency, N

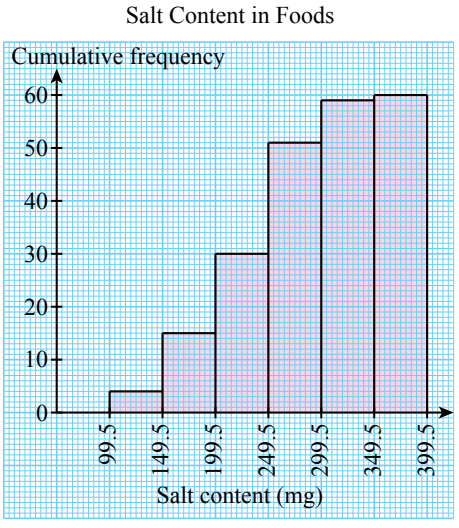
The median position, Q_2 = $\frac{1}{2}$ × total frequency, N

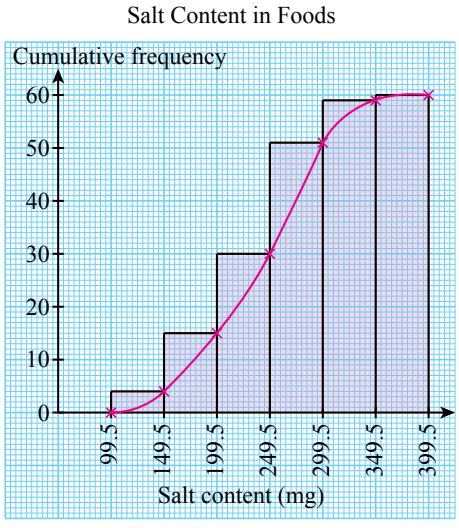
The third quartile position, Q_3 = $\frac{3}{4}$ × total frequency, N

Application & Career

A financial manager needs to be an expert in the features of market capital that involve financial assets such as stocks and bonds. Statistical method can be used to analyse the features of market capital through the stocks and bonds distributions.

Cumulative histogram and ogive can be constructed using cumulative frequency table. Cumulative histogram is constructed just like histogram, but the vertical axis is represented by cumulative frequency. By referring to Example 7, the cumulative histogram and the related ogive are as shown below.





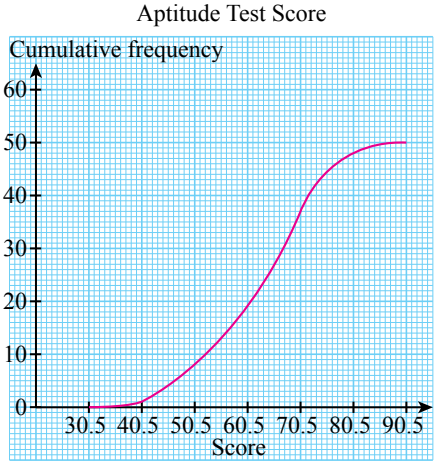
How is the construction of ogive related to the construction of cumulative histogram?

Percentile

We can analyse a large data more easily and effectively when we divide the data into small parts which is known as percentile. A **percentile** is a value that divides a set of data into 100 equal parts and is represented by $P_1, P_2, P_3, \ldots, P_{99}$.

Example 8

The ogive on the right shows the scores of an aptitude test obtained from candidates who are applying for a post in a company.



(a) Based on the ogive, find
  (i) the 10th percentile, P_{10}
  (ii) the 46th percentile, P_{46}

(b) Only those candidates who obtained 92nd percentile and above will be called for an interview. What is the minimum score required in order to be called for an interview?

(c) What is the percentage of the candidates who obtained a score of 57 and below?

Solution:

(a) (i) 10% of the total frequency $= \frac{10}{100} \times 50$

$= 5$

From the ogive, $P_{10} = 46.5$

(ii) 46% of the total frequency $= \frac{46}{100} \times 50$

$= 23$

From the ogive, $P_{46} = 63.5$

(b) 92% of the total frequency $= \frac{92}{100} \times 50$

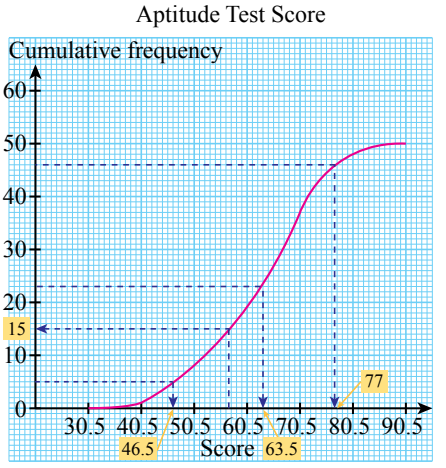
$= 46$

$P_{92} = 77$. Therefore, only candidates with a minimum score of 77 will be called for an interview.

(c) From the ogive,

$\frac{15}{50} \times 100 = 30\%$

Therefore, 30% of the candidates obtained a score of 57 and below.



Info Bulletin

25th percentile is also known as first quartile, 50th percentile as median and 75th percentile as third quartile.

Critical Mind

What is the difference between quartile and percentile?

Self Practice 7.1c

1. The frequency table on the right shows the marks of 100 pupils in an examination.

(a) Construct an ogive to represent the data.

(b) From your ogive, determine

(i) the first quartile

(ii) the median

(iii) the third quartile

Marks	Number of pupils
11 – 20	2
21 – 30	13
31 – 40	25
41 – 50	25
51 – 60	19
61 – 70	10
71 – 80	4
81 – 90	2

2. The frequency table on the right shows the length of the soles of 40 pupils.

(a) Construct an ogive to represent the data.

(b) Based on the ogive, find

(i) the 20th percentile, P_{20}

(ii) the 55th percentile, P_{55}

(iii) the 85th percentile, P_{85}

(c) What is the percentage of the pupils having a sole length of 24.6 cm and below?

Length of soles (cm)	Number of pupils
21.0 – 21.9	1
22.0 – 22.9	4
23.0 – 23.9	10
24.0 – 24.9	18
25.0 – 25.9	5
26.0 – 26.9	2

7.2 Measures of Dispersion

How to determine range, interquartile range, variance and standard deviation for grouped data?

Learning Standard

Determine range, interquartile range, variance and standard deviation as a measure to describe dispersion for grouped data.

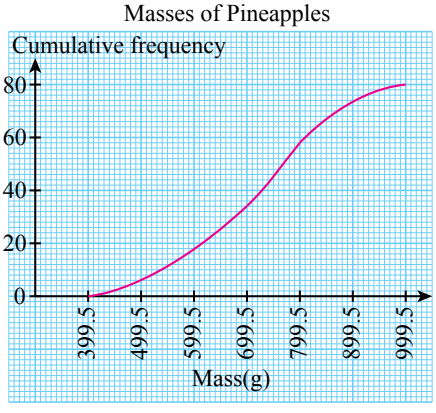
In Form 4, you have learnt ways to determine range, interquartile range, variance and standard deviation as a measure to describe dispersion for ungrouped data. In this section, we shall proceed to the measures of dispersion for grouped data.

Range and Interquartile Range

Example 9

Pak Hamidi had recorded the mass of pineapples that he harvested from his farm. The following frequency table and ogive show the data that he obtained. Determine the range and interquartile range for the data.

Mass (g)	Number of pineapples
400 – 499	6
500 – 599	12
600 – 699	16
700 – 799	24
800 – 899	14
900 – 999	8



MEMORY BOX

Interquartile range = $Q_3 - Q_1$

Smart TIPS

Interquartile range of a set of grouped data can be determined from ogive by finding Q_1 and Q_3 first.

Solution:

Range = midpoint of the highest class – midpoint of the lowest class

$$= \frac{900 + 999}{2} - \frac{400 + 499}{2}$$

$$= 949.5 - 449.5$$

$$= 500 \text{ g}$$

Difference between the heaviest pineapple and the lightest pineapple is 500 g.

From the ogive,

the position of Q_1:

$\frac{1}{4} \times 80 = 20$

$Q_1 = 614.5$

the position of Q_3:

$\frac{3}{4} \times 80 = 60$

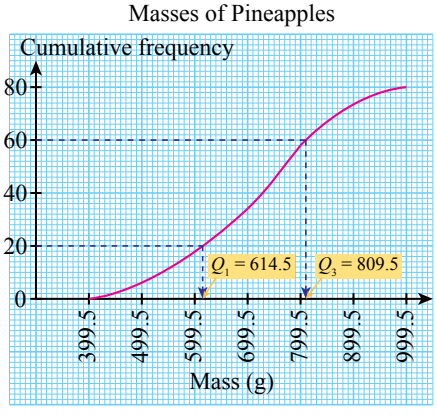
$Q_3 = 809.5$

Therefore, the interquartile range

$= 809.5 - 614.5$

$= 195$ g

Difference between the heaviest pineapple and the lightest pineapple that lies in the middle 50% of the distribution is 195 g.



Variance and Standard Deviation

Variance and standard deviation for a grouped data can be obtained using the following formulae.

Variance, $\sigma^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2$

Standard deviation, $\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$

where
x = midpoint of the class interval
f = frequency
$\bar{x}$ = mean of the data

MEMORY BOX

- Variance is the average of the square of the difference between each data and the mean.
- Standard deviation is a measure of dispersion relative to its mean, which is measured in the same unit of the original data.

Example 10

The frequency table below shows the volumes of water to the nearest litres, used daily by a group of families in a housing area. Calculate the variance and standard deviation of the data.

Volume of water (ℓ)	150 – 159	160 – 169	170 – 179	180 – 189	190 – 199	200 – 209
Number of families	8	12	15	24	20	16

Solution:

Volume of water (ℓ)	Frequency, f	Midpoint, x	fx	x^2	fx^2
150 – 159	8	154.5	1 236	23 870.25	190 962
160 – 169	12	164.5	1 974	27 060.25	324 723
170 – 179	15	174.5	2 617.5	30 450.25	456 753.75
180 – 189	24	184.5	4 428	34 040.25	816 966
190 – 199	20	194.5	3 890	37 830.25	756 605
200 – 209	16	204.5	3 272	41 820.25	669 124
	$\sum f = 95$		$\sum fx = 17\ 417.5$		$\sum fx^2 = 3\ 215\ 133.75$

Mean, $\bar{x} = \frac{\sum fx}{\sum f}$

$= \frac{17\ 417.5}{95}$

$= 183.34\ \ell$

Variance, $\sigma^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2$

$= \frac{3\ 215\ 133.75}{95} - \left(\frac{17\ 417.5}{95}\right)^2$

$= 229.1856$

$= 229.19\ \ell^2$ (correct to 2 decimal places)

Standard deviation, $\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$

$= \sqrt{229.1855956}$

$= 15.1389$

$= 15.14\ \ell$ (correct to 2 decimal places)

Checking Answer

1. Press [MODE] [MODE]
   Display [SD 1 | REG 2 | BASE 3]
   Choose 1
2. Enter (midpoint), press [SHIFT] [,] (frequency) [M+], and repeat for the subsequent values.
3. Press [AC] [SHIFT] [2]
   Display [$\bar{x}$ 1 | $x\sigma n$ 2 | $x\sigma n{-}1$ 3]
   Press 1 for mean:
   Display [183.3421053]
   Press 2 for standard deviation:
   Display [15.13887696]

Self Practice 7.2a

1. The frequency table below shows the electricity bills of apartment units for a certain month.

Electricity bill (RM)	30 – 49	50 – 69	70 – 89	90 – 109	110 – 129
Number of apartment units	4	9	11	15	13

Construct an ogive for the data and hence, calculate the range and interquartile range. Explain the meaning of the range and interquartile range obtained.

2. Calculate the variance and standard deviation of each of the following data. Give your answer correct to two decimal places.

(a)

Time (minutes)	1 – 2	3 – 4	5 – 6	7 – 8	9 – 10	11 – 12
Frequency	15	20	28	35	30	24

(b)

Distance (m)	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80
Frequency	5	8	13	20	22	21	11

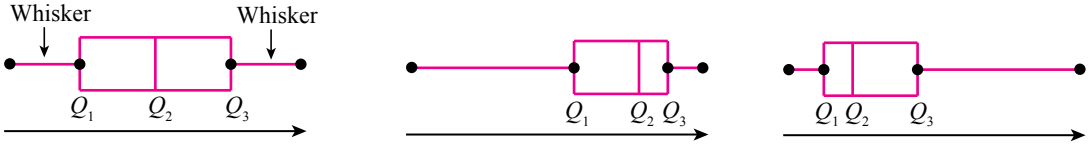
How to construct and interpret a box plot for a set of grouped data?

Learning Standard

Construct and interpret a box plot for a set of grouped data.

You have learnt that a box plot is a method to display a group of numerical data graphically based on the five number summary of data. They are the minimum value, first quartile, median, third quartile and maximum value. Similar to the histrogram and frequency polygon, the shape of a distribution can also be identified through the box plot.

(a) (b)(i) (b)(ii)



Symmetric distribution **Left-skewed distribution** **Right-skewed distribution**

(a) The median lies in the middle of the box and the whiskers are about the same length on both sides of the box.
(b) The median cuts the box into two different sizes.
  (i) If the left side of the box is longer, then the data distribution is left-skewed.
  (ii) If the right side of the box is longer, then the data distribution is right-skewed.

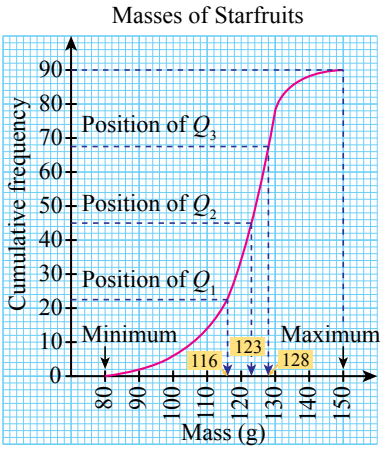
Left whisker and right whisker represent the score outside of the median. If the box is divided into the same size but the left whisker is longer than the right whisker, then the data distribution is left-skewed, and vice versa.

Example 11

The ogive on the right shows the masses in g, of 90 starfruits.

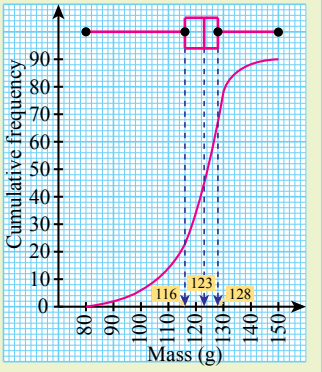
(a) Construct a box plot based on the ogive.

(b) Hence, state the distribution shape of the data.

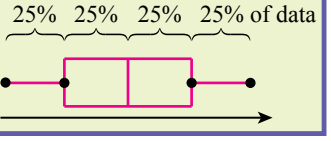


Info Bulletin

Ogive and box plot on the same graph:



Info Bulletin

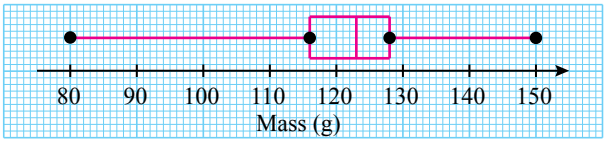


Solution:

(a) From the ogive:

- Minimum value = 80
- Maximum value = 150
- Position of Q_1: $\frac{1}{4} \times 90 = 22.5$
  $Q_1 = 116$
- Position of Q_2: $\frac{1}{2} \times 90 = 45$
  $Q_2 = 123$
- Position of Q_3: $\frac{3}{4} \times 90 = 67.5$
  $Q_3 = 128$

Box plot:



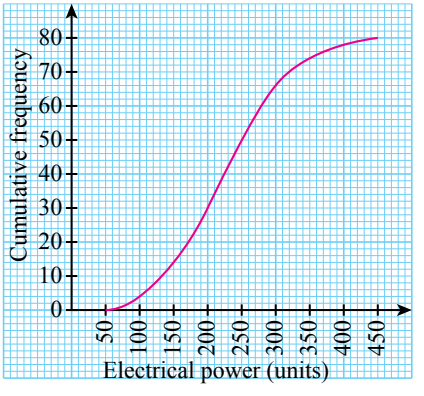
(b) The distribution of the data is skewed to the left because the left side of the box plot is longer than the right side of the box plot.

Self Practice 7.2b

1. The ogive on the right shows the number of units of electrical power, consumed by 80 households in a particular month.

(a) Construct a box plot based on the ogive.

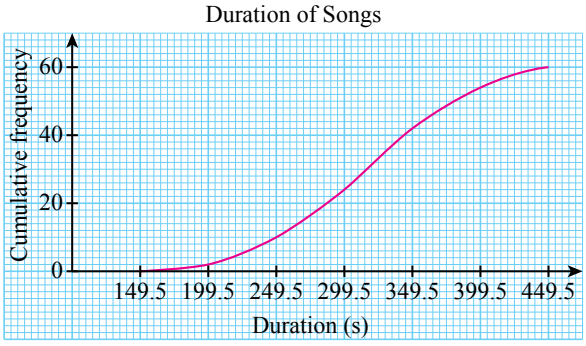
(b) Hence, state the distribution shape of the data.



2. The ogive on the right shows the duration in seconds, of 60 songs aired by a radio station at a certain time.

(a) Construct a box plot based on the ogive.

(b) Hence, state the distribution shape of the data.



How to compare and interpret two or more sets of grouped data based on measures of dispersion?

Learning Standard

Compare and interpret two or more sets of grouped data, based on measures of dispersion hence make conclusion.

Example 12

A botanist sowed 40 samples of hibiscus seeds using two different hybrids, A and B. The diameters of both hybrids are measured under close guard to develop an extra-large hibiscus. The following frequency table shows the diameters of petals for hybrid A and B.

Diameter (cm)	13.0 – 13.4	13.5 – 13.9	14.0 – 14.4	14.5 – 14.9	15.0 – 15.4
Hybrid A	4	8	9	10	9
Hybrid B	9	10	8	6	7

Based on the mean and standard deviation, determine which hybrid produces larger and more consistent petals. Justify your answer.

Solution:

For hibiscus of hybrid A,

Diameter (cm)	Frequency, f	Midpoint, x	fx	x^2	fx^2
13.0 – 13.4	4	13.2	52.8	174.24	696.96
13.5 – 13.9	8	13.7	109.6	187.69	1 501.52
14.0 – 14.4	9	14.2	127.8	201.64	1 814.76
14.5 – 14.9	10	14.7	147	216.09	2 160.9
15.0 – 15.4	9	15.2	136.8	231.04	2 079.36
	$\sum f = 40$		$\sum fx = 574$		$\sum fx^2 = 8\ 253.5$

Mean, $\bar{x} = \frac{574}{40}$

$= 14.35$ cm

Standard deviation, $\sigma = \sqrt{\frac{8\ 253.5}{40} - 14.35^2}$

$= \sqrt{0.415}$

$= 0.64$ cm

For hibiscus of hybrid B,

Diameter (cm)	Frequency, f	Midpoint, x	fx	x^2	fx^2
13.0 – 13.4	9	13.2	118.8	174.24	1 568.16
13.5 – 13.9	10	13.7	137	187.69	1 876.9
14.0 – 14.4	8	14.2	113.6	201.64	1 613.12
14.5 – 14.9	6	14.7	88.2	216.09	1 296.54
15.0 – 15.4	7	15.2	106.4	231.04	1 617.28
	$\sum f = 40$		$\sum fx = 564$		$\sum fx^2 = 7\ 972$

Mean, $\bar{x}$

$= \frac{564}{40}$

$= 14.1$ cm

Standard deviation, σ

$= \sqrt{\frac{7\ 972}{40} - 14.1^2}$

$= \sqrt{0.49}$

$= 0.7$ cm

Hybrid A produces larger petals because the mean is larger than hybrid B (14.35 cm $>$ 14.1 cm) and the smaller standard deviation (0.64 cm $<$ 0.7 cm) shows that the diameter of the petals is more consistent.

My Malaysia

Tunku Abdul Rahman Putra Al-Haj declared the hibiscus as The National Flower in 1960. The five petals of the flower represent the five principles of *Rukun Negara*.

Self Practice 7.2c

1. A ball manufacturing factory needs to regulate the internal air pressure in psi, of the produced ball before being marketed. The frequency table below shows the internal air pressures of 50 ball samples taken from machine P and machine Q.

Air pressure (psi)	8.0 – 8.9	9.0 – 9.9	10.0 – 10.9	11.0 – 11.9	12.0 – 12.9	13.0 – 13.9
Machine P	7	11	13	12	5	2
Machine Q	1	3	5	20	18	3

The factory specified that the internal air pressure of a ball should be between 11.3 psi to 11.7 psi. Which machine shows better performance in terms of air pressure accuracy?

2. The frequency table below shows the lifespans in years, of brand X and brand Y batteries.

Lifespan (years)	0 – 0.9	1.0 – 1.9	2.0 – 2.9	3.0 – 3.9	4.0 – 4.9
Brand X battery	4	10	17	20	9
Brand Y battery	10	21	15	8	6

By using suitable measures, determine which brand of battery is better and lasts longer.

How to solve problems involving measures of dispersion for grouped data?

Learning Standard

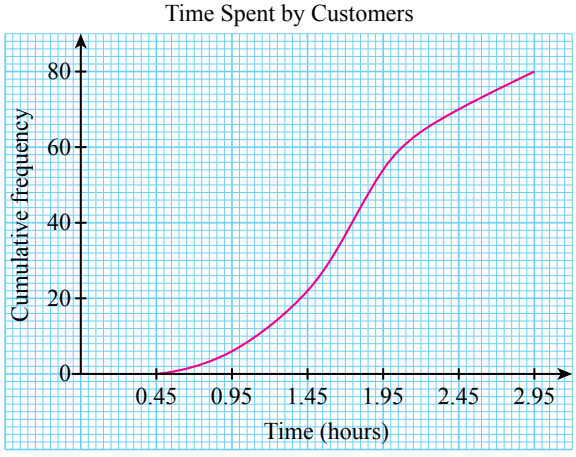
Solve problems involving measures of dispersion for grouped data.

Example 13

A survey on the duration of time in hours, spent by customers to buy goods in a supermarket is carried out. The results of the survey are shown in the ogive on the right.

(a) Construct a frequency table for the time taken by the customers to buy goods in the supermarket using the classes 0.5 – 0.9, 1.0 – 1.4, 1.5 – 1.9, 2.0 – 2.4 and 2.5 – 2.9.

(b) Hence, estimate the mean and standard deviation of the data.



Solution:

Understanding the problem

Determine the mean and standard deviation from the ogive.

Devising a strategy

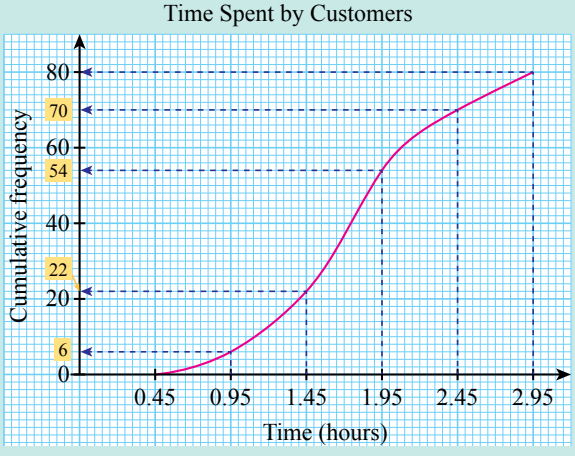
(a) Construct the frequency table from the ogive.

(b) Calculate the mean and standard deviation using formula.

Implementing the strategy

(a)

Time (hours)	Number of customers	
0.5 – 0.9	6	6 – 0 = 6
1.0 – 1.4	16	22 – 6 = 16
1.5 – 1.9	32	54 – 22 = 32
2.0 – 2.4	16	70 – 54 = 16
2.5 – 2.9	10	80 – 70 = 10



(b)

Time (hours)	Frequency, f	Midpoint, x	fx	x^2	fx^2
0.5 – 0.9	6	0.7	4.2	0.49	2.94
1.0 – 1.4	16	1.2	19.2	1.44	23.04
1.5 – 1.9	32	1.7	54.4	2.89	92.48
2.0 – 2.4	16	2.2	35.2	4.84	77.44
2.5 – 2.9	10	2.7	27	7.29	72.9
	$\sum f = 80$		$\sum fx = 140$		$\sum fx^2 = 268.8$

Mean, $\bar{x} = \frac{140}{80}$

$= 1.75$ hours

Standard deviation, $\sigma = \sqrt{\frac{268.8}{80} - 1.75^2}$

$= \sqrt{0.2975}$

$= 0.55$ hours

Making a conclusion

(a)

Time (hours)	Number of customers
0.5 – 0.9	6
1.0 – 1.4	16
1.5 – 1.9	32
2.0 – 2.4	16
2.5 – 2.9	10

(b) Mean, $\bar{x} = 1.75$ hours

Standard deviation, $\sigma = 0.55$ hours

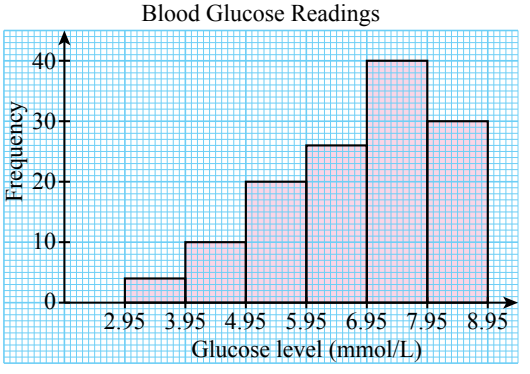
Self Practice 7.2d

1. The histogram on the right shows the blood glucose readings taken from a group of patients in a clinic.

(a) Based on the histogram, is the distribution symmetrical? Give your reason.

(b) Calculate the mean and standard deviation of the blood glucose readings.

(c) Compare the standard deviation of the blood glucose readings between 6.0 mmol/L and 8.9 mmol/L with the standard deviation in (b). Justify your answer.



2. The table below shows the results of statistical analysis for the price in RM, of 10 kg of rice in supermarket P and supermarket Q. The number of data taken is 20 rice bags from each supermarkets respectively.

Supermarket	Mean	Standard deviation	Minimum value	First quartile	Median	Third quartile	Maximum value
P	32	5.62	26	30	32	34	40
Q	32	4.05	26	32	34	34	40

(a) State the mean and range of the price of the rice in both supermarkets.
(b) The price distribution of the rice in which market is more symmetric? Explain your answer.
(c) Discuss about the median and interquartile range of the two data.

How to design and conduct a mini-project involving statistical investigations?

Learning Standard

Design and conduct a mini-project involving statistical investigations based on measures of central tendency and measures of dispersion and hence interpret and communicate research findings.

PROJECT

The statistical data collected by the National Health and Morbidity Survey (NHMS) shows that more Malaysians are becoming obese with the rate of one in two adults suffering from overweight. Other surveys also found that overweight and obesity among school pupils make up 30% of the population. Obesity increases the chances of developing health problems such as diabetes, heart disease and stroke.

Your school has decided to launch "Activate Your Life" campaign with the aim to create awareness among pupils about obesity and motivate them to adopt healthy lifestyle. Your mathematics teacher wishes to display the health level of pupils on the school bulletin board according to their gender.

Title: Health Level of Pupils

Material: Measuring tape, weighing scale

Procedure:

1. Each group will investigate the health level of pupils based on their gender by using the Body Mass Index (BMI). Divide the class into five groups where each group conducts survey to Form 1, 2, 3, 4 and 5 pupils. Fix the equal number of respondents according to gender from each form.

Smart TiPS

You can carry out the research outside the teaching and learning (PdP) session.

2. Each group is asked to prepare a project report as part of the learning in the class. The report needs to cover the following aspects:

(a) **Survey**
Generate suitable questions such as gender, height, mass and the number of hours spent in sport activities by the respondents within a week as part of the data collection process.

(b) **Data collection method**
Choose a collection method to obtain the data.
Choose your respondents randomly.

(c) **Data organisation method**
Construct a frequency table to organise your data.
Choose an appropriate class interval for each data.

(d) **Graphical representations**
Present your data using histogram, frequency polygon and other suitable representations.

Smart TiPS

You can organise the data and construct the graphical representation using electronic spreadsheet.

(e) **Data analysis**

(i) Calculate the suitable measures of central tendency and measures of dispersion for each of your data.

(ii) Calculate the Body Mass Index (BMI) of each pupil using the following formula.

$$\text{BMI} = \frac{\text{Mass (kg)}}{\text{Height (m)} \times \text{Height (m)}}$$

(iii) The table below shows the BMI according to age for a teenage boy.

Age	Underweight	Normal	Overweight	Obese
13	$\leqslant 14.8$	14.9 – 20.8	20.9 – 24.8	> 24.8
14	$\leqslant 15.4$	15.5 – 21.8	21.9 – 25.9	> 25.9
15	$\leqslant 15.9$	16.0 – 22.7	22.8 – 27.0	> 27.0
16	$\leqslant 16.4$	16.5 – 23.5	23.6 – 27.9	> 27.9
17	$\leqslant 16.8$	16.9 – 24.3	24.4 – 28.6	> 28.6

The table below shows the BMI according to age for a teenage girl.

Age	Underweight	Normal	Overweight	Obese
13	$\leqslant 14.8$	14.9 – 21.8	21.9 – 26.2	> 26.2
14	$\leqslant 15.3$	15.4 – 22.7	22.8 – 27.3	> 27.3
15	$\leqslant 15.8$	15.9 – 22.5	22.6 – 28.2	> 28.2
16	$\leqslant 16.1$	16.2 – 24.1	24.2 – 28.9	> 28.9
17	$\leqslant 16.3$	16.4 – 24.8	24.9 – 29.3	> 29.3

Source: World Health Organization (WHO), 2007

Based on the BMI table and the data collected, determine the percentage of pupils that are under the category 'Underweight' and 'Obese' by constructing an ogive.

(f) **Description and conclusion**

(i) Interpret your findings of the research. Make conclusions about the physical state of pupils according to their gender for each form.

(ii) Suggest follow-up actions that need to be taken by the pupils who are underweight, overweight and obese.

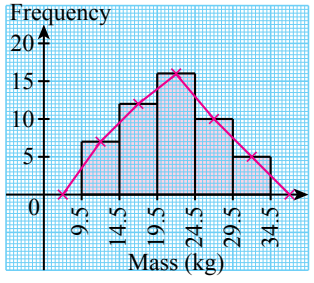
3. Write down your findings of the research on the cardboards and paste it on your school bulletin board.

Summary Arena

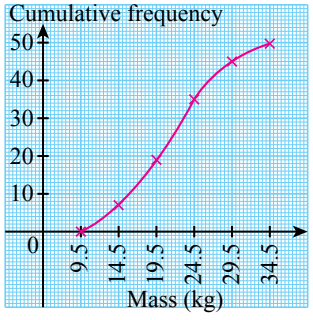
MEASURES OF DISPERSION FOR GROUPED DATA

Data Representation

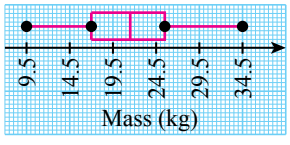
Histogram and frequency polygon:



Ogive:



Box plot:



Measures of Dispersion

Range
= Midpoint of the highest class − Midpoint of the lowest class

Interquartile range
$= Q_3 - Q_1$
(The values of Q_3 and Q_1 are determined from ogive)

Percentile

Variance, $\sigma^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2$

Standard deviation, $\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$

Reflection

At the end of this chapter, I can

construct histogram and frequency polygon for a set of grouped data.		
compare and interpret the dispersions of two or more sets of grouped data based on histogram and frequency polygon, hence make conclusion.		
construct an ogive for a set of grouped data and determine the quartiles.		
determine range, interquartile range, variance and standard deviation as a measure to describe dispersion for grouped data		
construct and interpret a box plot for a set of grouped data.		
compare and interpret two or more sets of grouped data, based on measures of dispersion hence make conclusion.		
solve problems involving measures of dispersion for grouped data.		
design and conduct a mini-project involving statistical investigations based on measures of central tendency and measures of dispersion and hence interpret and communicate research findings.		

MINI PROJECT

You are required to investigate the population distribution of Malaysia, Indonesia and Singapore from 1990 to 2019. You can obtain the population data by scanning the QR code on the right.

Then, organise the data in frequency table using the appropriate class interval. Construct a suitable data representation to see the data distribution.

For the data of each country, obtain the values of the measures of central tendency and the measures of dispersion. By using the value of suitable measure, compare the populations in these three countries from the aspect of total population and population dispersion. Make a conclusion for the population distribution and relate it with the population density in each country.

Scan the QR code or visit bit.do/MPChap7 to obtain the population data.

Extensive Practice

Scan the QR code or visit bit.do/QuizE07 for interactive quiz

UNDERSTAND

1. For each of the following class intervals, determine the lower limit, upper limit, midpoint, lower boundary and upper boundary.

(a)

Length (m)	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34

(b)

Volume (cm^3)	25.0 – 25.9	26.0 – 26.9	27.0 – 27.9	28.0 – 28.9	29.0 – 29.9

(c)

Mass (g)	0 – 0.24	0.25 – 0.49	0.50 – 0.74	0.75 – 0.99

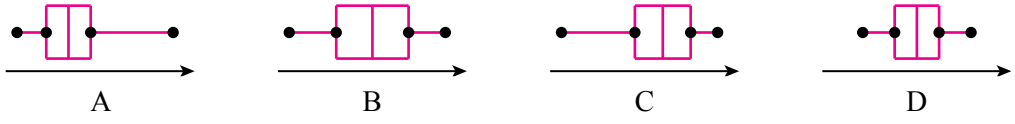
2. Identify whether the following descriptions is for a histogram, a frequency polygon or an ogive.

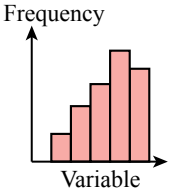
(a) Graph that represents cumulative frequency of classes in a frequency distribution. ☐

(b) Displays data using side by side bars. The height of the bar is used to represent the class frequency. ☐

(c) Displays data using straight lines that connect the midpoints of the class interval. Frequency is represented by the height of these midpoints. ☐

3. The box plot below shows the distribution shapes of data. Match the following histograms with the corresponding box plot in the space provided.



(a)


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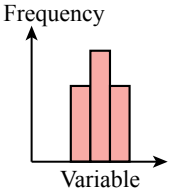
(b)


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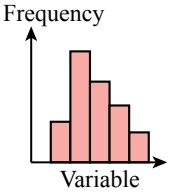
(c)


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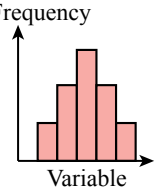
(d)


☐

MASTERY

4. The data on the right shows the heights in cm, of 30 pupils in Form 5.

(a) Organise the data by completing the frequency table below. Then, draw a histogram of the data using a suitable scale.

Height (cm)	Lower boundary	Upper boundary	Tally	Frequency
145 – 149				
150 – 154				

146	163	156
152	174	156
178	151	148
166	154	150
164	157	171
168	159	170
163	157	161
167	162	157
166	160	155
168	158	162

(b) Construct a new frequency table by rearranging the class intervals beginning with 145 – 148 cm, 149 – 152 cm, 153 – 156 cm and so forth. Hence, construct a histogram to display the data.

(c) Compare the distribution shapes of the two histograms. In your opinion, what is the conclusion that can be made from this comparison?

5. The frequency table below shows the time spent watching television in a week by 30 families.

Time (hours)	2 – 4	5 – 7	8 – 10	11 – 13	14 – 16	17 – 19	20 – 22
Number of families	8	9	6	4	2	0	1

(a) On the same graph, construct a histogram and frequency polygon of the data using a suitable scale.

(b) Comment on the distribution shape of the data displayed.

6. The frequency table below shows the Mathematics test marks of a group of pupils.

Marks	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89	90 – 99
Number of pupils	4	8	12	10	9	7

Draw an ogive of the data and calculate

(a) the range,

(b) the interquartile range,

(c) the 40^{th} percentile and 80^{th} percentile for the test marks.

CHALLENGE

7. Khuzairi is a dairy cow raiser. He rears 130 dairy cows in his farm. The frequency table below shows the volumes in litres, of milk produced by his cows in a certain week.

Volume of milk (litres)	5 – 10	11 – 16	17 – 22	23 – 28	29 – 34	35 – 40
Number of dairy cows	15	28	37	26	18	6

(a) Construct a cumulative histogram of the data.

(b) On the same graph in (a), construct an ogive. Hence, estimate the interquartile range of the distribution.

8. The frequency table below shows the blood pressure readings taken from a group of patients before and after taking a dose of a type of medicine in lowering blood pressure.

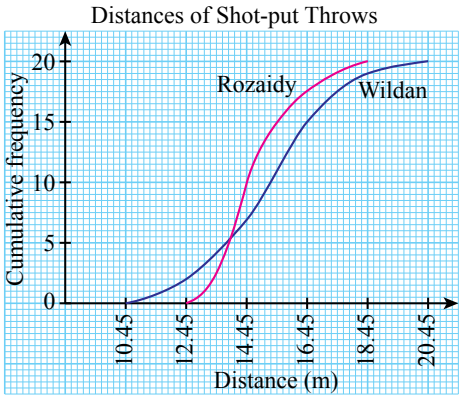
Systolic blood pressure (mmHg)	120 – 134	135 – 149	150 – 164	165 – 179
Before	4	7	8	6
After	9	8	7	1

Calculate the mean and standard deviation of the data. Is the medicine effective in lowering down the blood pressure of the group of patients after taking a dose of the medicine? Justify your answer.

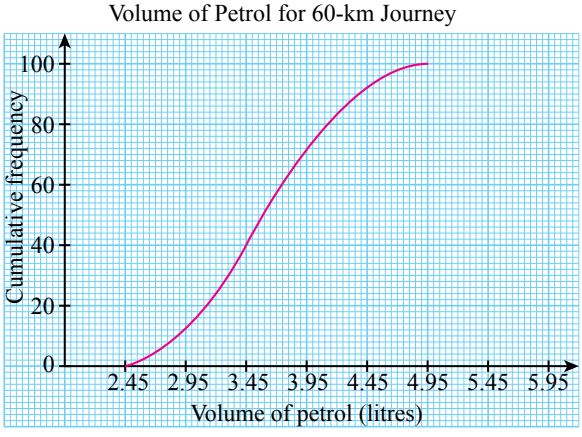
9. The ogive on the right shows the distances of shot-put throws obtained by Rozaidy and Wildan in a training session.

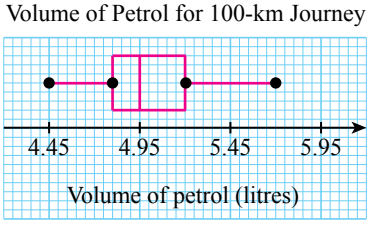
(a) Calculate the percentage of the throwing distances that exceeds 15.45 m obtained by Rozaidy and Wildan.

(b) Based on the median and the third quartile of both performances, determine who perform better during the training session.



10. Volumes of petrol consumed by 100 cars were recorded. The ogive shows the volume of petrol consumed for a 60-km journey and the box plot shows the volume of petrol consumed for a 100-km journey.





(a) Redraw the ogive for the 60-km journey. On the same graph, draw an ogive for the volume of petrol consumed for the 100-km journey.

(b) If a car uses 3.7 litres petrol for the 60-km journey, calculate the volume of petrol consumed for the 100-km journey. Justify your answer.

EXPLORING MATHEMATICS

Instructions:

(i) Do this activity in a small group.

(ii) Each group answer the question in the activity worksheet (scan the QR code).

(iii) After completing the worksheet, each group needs to construct a mind map that summarises the distribution shape and the suitable measures to describe the data.

(iv) Present the outcome of your group. The best presentation will be displayed at the Mathematics corner in your class.

Scan the QR code or visit bit.do/EMChap7 to perform this activity.

CHAPTER 8

Mathematical Modeling

What will you learn?

- Mathematical Modeling

Why study this chapter?

Mathematical modeling is used in various disciplines in the real world. Engineers use mathematical modeling to analyse the traffic flow on a bridge. Telecommunication companies use mathematical modeling to determine the charges for the calls that consumers make. Scientists also use mathematical modeling to predict the trends in population growth and the spread of infectious diseases to ensure the well-being of human race.

Do you know?

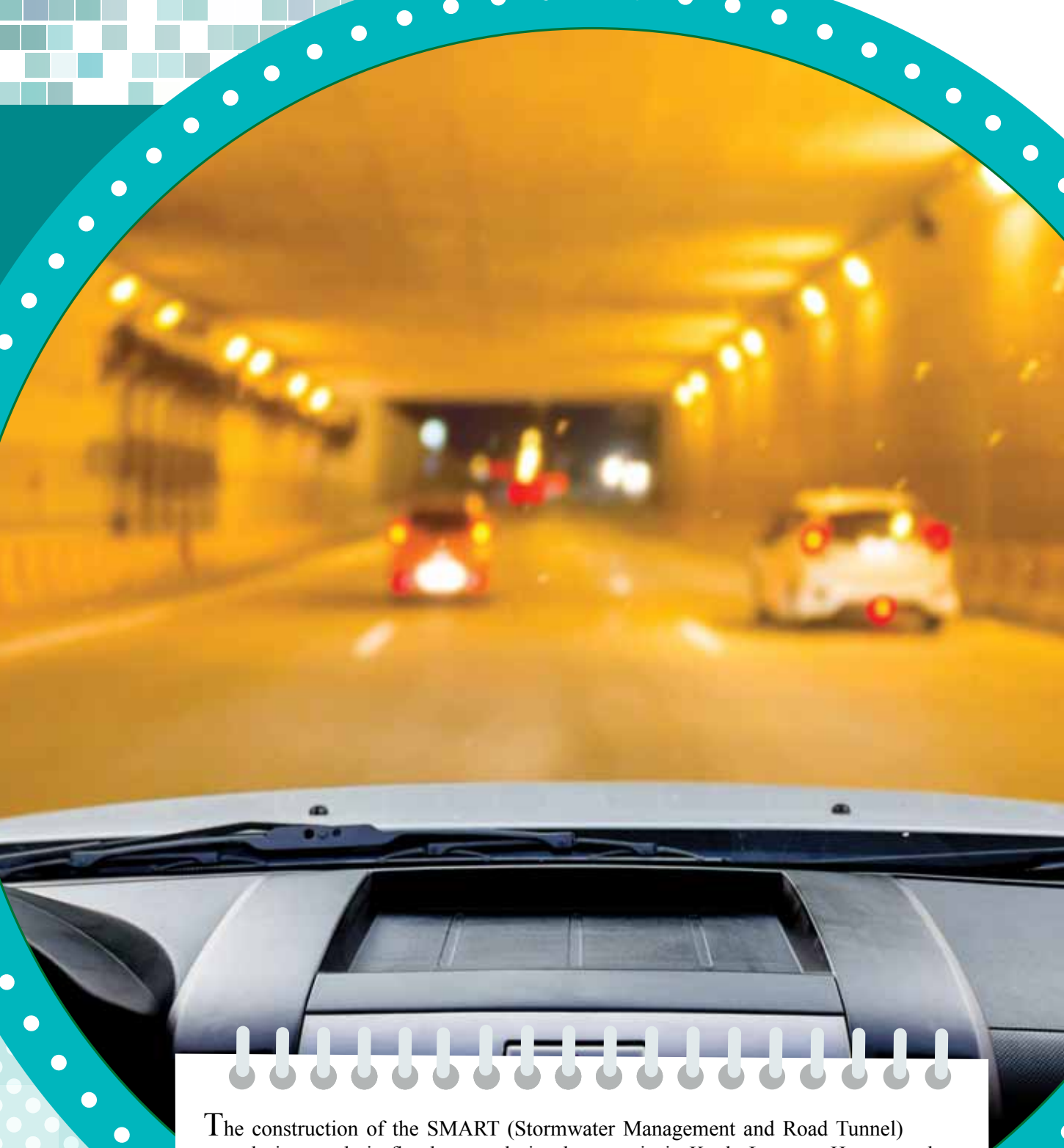
The Covid-19 pandemic has hit the world in 2019. In this regard, the Malaysian government implemented the Movement Control Order (MCO) in an effort to flatten the epidemiological curve of Covid-19 cases in our country. Mathematical modeling can be used to make epidemic predictions.

For more information:

bit.do/DoYouKnowChap8

WORD BANK

exponent	*eksponen*
exponential function	*fungsi eksponen*
quadratic	*kuadratik*
linear	*linear*
mathematical modeling	*pemodelan matematik*



The construction of the SMART (Stormwater Management and Road Tunnel) tunnel aims to drain flood water during heavy rain in Kuala Lumpur. However, the tunnel also has an additional function in reducing traffic congestion. The unique design and innovative way of operating the tunnel presented challenges to the engineers at the beginning of its construction. Do you know that mathematical modeling was used to demonstrate the four different operation modes of the SMART tunnel?

8.1 Mathematical Modeling

Learning Standard

Explain mathematical modeling.

What is mathematical modeling?

The world around us is filled with various important questions that are not answered. For example,

What is the effect of rising sea levels on the coastal regions of Malaysia?

How much will it cost to go to college or university in 10 years?

Will the population in Malaysia surpass 40 million?

Is it possible to identify the personality trait of cybercrime victims by studying the behaviours of the web surfers?

Answers to these questions are constantly being studied by researchers. Will they be able to find the answers? Maybe. The only thing one can say with certainty is that any attempt to find the solution requires the use of mathematics, most likely through the creation, application and refinement of **mathematical models**.

A mathematical model is a mathematical relation that describes the real-world situation. For example, the formula $I = Prt$ is a relation between simple interest and the other three quantities, that is, the principal, the interest rate and the time.

In this chapter, you will be introduced to the process of constructing mathematical models, which is called **mathematical modeling**. In mathematical modeling, a real-world problem will be translated into a mathematical problem. We then solve the mathematical problem and interpret its solution in terms of the real-world problem.

A **mathematical model** is a representation of a system or scenario that is used to gain qualitative and/or quantitative understanding of some real-world problems and to predict future behaviour.

Consider the following problems.

1 The population of town A is 40 000. It is known that 45% of the residents in the town practise the habit of recycling used plastic drinking bottles. If each person in the town uses 5 plastic drinking bottles per week, how many bottles are recycled each week in the town?

This type of question is often asked in Mathematics books to reinforce the concept of percentage. This is an example of a problem solving question. The problem explicitly gives us all the information we need. We only need to determine the appropriate computations to arrive at one correct answer. This kind of problem solving question can be used to help us understand a particular mathematical concept and reinforce important mathematical skills.

Let us look at the following question.

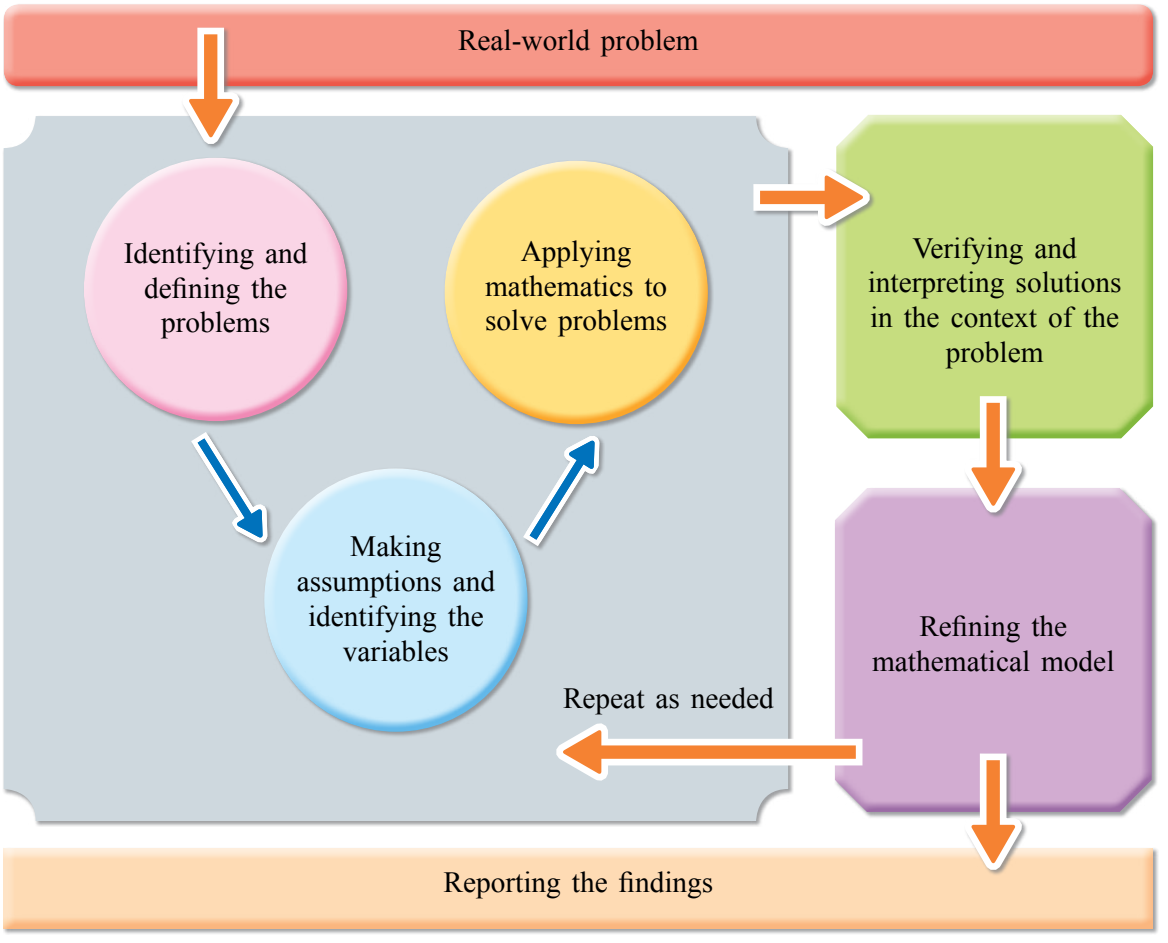
2 How long does it take to clean up an oil spill in the ocean?

The second question is quite different. We do not have enough information to answer this question. It is an open-ended question. We usually do not have complete information when trying to solve real-world problems. These real-world problems demand us to use both mathematics knowledge and creativity to solve them. In this case, mathematical modeling is perfect for solving such open-ended questions.

There are a few important components in mathematical modeling:

1. Identifying and defining the problems
2. Making assumptions and identifying the variables
3. Applying mathematics to solve problems
4. Verifying and interpreting solutions in the context of the problem
5. Refining the mathematical model
6. Reporting the findings

The modeling process can be simplified as shown in the diagram below.



Let us look at one mathematical modeling example that involves all the above components.

Real-world Problem

The grocery store which is far from your house sells a pack of 10 kg rice at a lower price than the one closer to your house. Is it worth the drive for the cheaper deal?

Identifying and defining the problems

- The price of a pack of 10 kg rice and the cost of petrol.
- Determine the distance from the two stores to your house respectively.
- Find the information needed from the internet, for example the price of a pack of 10 kg rice from each store, the distance of each store from your house, the current petrol price, the car's petrol consumption rate and so on.

Making assumptions

- The route between the house, store A and store B is a perfect straight line
- Driving car to the stores
- Purchasing the same number and brand of rice at store A and store B

Smart TIPS

The assumptions made usually start with the simplest variables. Once the problem is solved, more complex assumptions can be considered.

Identifying the variables

Let
- s = the distance between store A and store B
- P_1 = the price of a pack of 10 kg rice at store A
- P_2 = the price of a pack of 10 kg rice at store B
- m = the petrol consumption rate of the car in km per litre
- n = the number of packs of 10 kg rice to be purchased
- H = the current petrol price in RM per litre
- S = the difference in price in RM for purchasing the pack of 10 kg rice at store B as compared to store A
- T = the difference in the cost of petrol in RM for driving to store B as compared to store A

MEMORY BOX

A variable is a quantity with an unknown value.

Applying mathematics to solve problems

The price of a pack of 10 kg rice

Store A: RM25.95
Store B: RM23.99
Let the quantity of 10 kg rice to be purchased be 2 packs.

The price to be paid

Store A: 2 × RM25.95 = RM51.90
Store B: 2 × RM23.99 = RM47.98

Therefore, by purchasing the rice at store B, one can save
RM51.90 – RM47.98 = RM3.92

Let the distance between store A and store B be 6 km,
the petrol consumption rate of a car be approximately 17.6 km per litre,
the current petrol price be RM2.08 per litre.

Petrol needed for the 6 km route = $\frac{6 \text{ km}}{17.6 \text{ km/litre}} = 0.341 \text{ litre}$

Cost of petrol for the 12 km (two ways) route = $0.341 \text{ litre} \times \frac{\text{RM}2.08}{\text{litre}} \times 2 = \text{RM}1.42$

Therefore, we save RM3.92 – RM1.42 = RM2.50 by purchasing two packs of 10 kg rice at store B.

Verifying and interpreting solutions in the context of the problem

The resulting mathematical model is as follows.

$S = (P_1 - P_2) \times n$

$T = \frac{s}{m} \times H \times 2$

If $S > T$, then it is sensible to drive to store B to purchase rice and save more money.
If $S \leqslant T$, then we should not drive to store B to purchase rice.

Example of using the model:

$S = (P_1 - P_2) \times n$
$= (25.95 - 23.99) \times 2$
$= \text{RM}3.92$

$T = \frac{s}{m} \times H \times 2$
$= \frac{6}{17.6} \times 2.08 \times 2$
$= \text{RM}1.42$

When verifying and interpreting solutions, we consider:
- Are the solutions able to solve the problem?
- Is it rational to translate the solutions to real-world situation?

Since $S > T$, you save more money by purchasing rice at store B. Hence it is sensible to drive to store B. Thus, the mathematical model is able to solve the problem.

From the constructed mathematical model, further exploration can be made.

1. If all the variables remain the same, how far can store B be located so that it is an optimum choice?

$$(P_1 - P_2) \times n > \frac{s}{m} \times H \times 2$$
$$(25.95 - 23.99) \times 2 > \frac{s}{17.6} \times 2.08 \times 2$$
$$s < 16.6 \text{ km}$$

This means that if the distance between the two stores is less than 16.6 km, then it is **sensible to drive to store B** to purchase rice.

Info Bulletin

Optimum means the best or the most profitable.

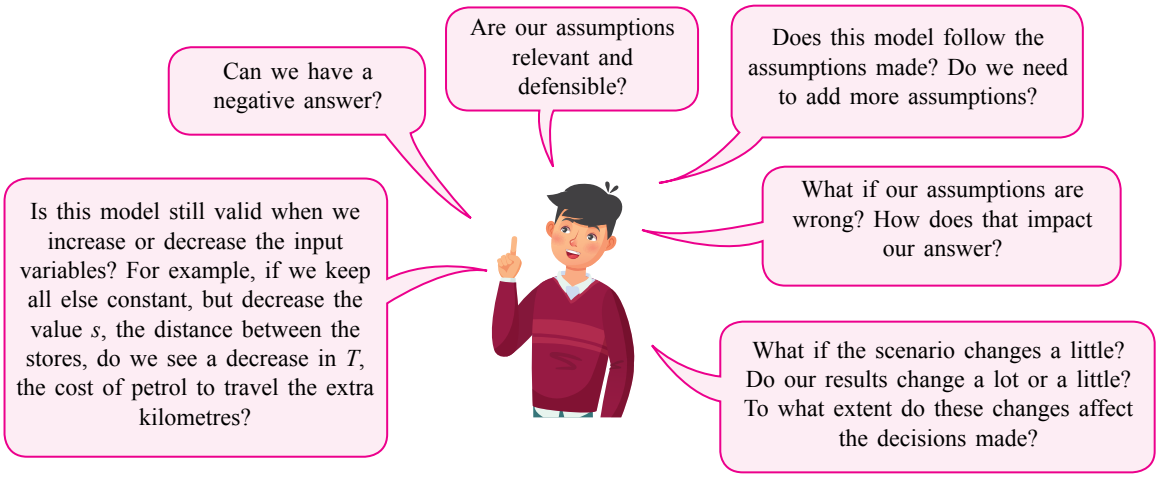
2. What is the maximum price of a pack of 10 kg rice at store B that will attract a person drive to store B to purchase it?

$$(P_1 - P_2) \times n > \frac{s}{m} \times H \times 2$$
$$(25.95 - P_2) \times 2 > \frac{6}{17.6} \times 2.08 \times 2$$
$$P_2 < \text{RM}25.24$$

This means that if the price of a pack of 10 kg rice at store B is less than RM25.24, then it is worth the trip.

Therefore, the above mathematical model allows us to not only answer the question of "is it worth it" for the given situation, but also allows us to determine other factors that support or oppose the decision to drive further to purchase the rice.

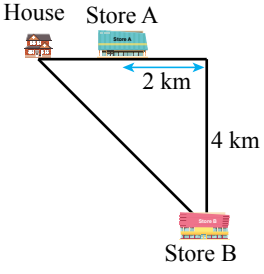
We now reflect on the model and the questions that arise as follows.



These reflections enable us to think of the needs to refine the mathematical model.

Refining the mathematical model

The assumption made regarding the route between the house, store A and store B is a straight line needs to be revised. If the route between the three places is not a straight line, how does this impact our model? If new assumptions are being made, then the model needs to be revised to reflect this change.



Reporting the findings

Use symbols and diagrams when necessary to report the findings. The symbols and diagrams will depict the whole modeling process until it leads to results. Every model has strengths and weaknesses. What is important is that the model identifies those strengths and weaknesses in the report.

For example,

- the value of a person's time is not considered in this model. Is it worth driving an extra 12 km for someone to save RM2.50? According to this model, saving 5 sen is considered worthwhile.
- the model did not account for any environmental considerations. Is it environmentally responsible to drive an extra 12 km just to save RM2.50? This model did not account for environmental effort such as efforts to reduce carbon emissions.

We have now seen how the mathematical modeling process works by one example as shown above. We should emphasise that we cannot use this example as a template to solve other problems. In solving problems through mathematical modeling, every individual or every group has different ideas, skills and perceptions. Open-ended questions such as these may even generate completely different mathematical models which are valid yet result in different answers.

Example 1

Jeremy saves RM4 000 at Bank Bunga Raya with a simple interest rate of 1.5% per annum. Jeremy wants to buy a computer worth RM4 455 with his savings. How long does Jeremy need to save?

(a) Identify and define the problem.

(b) Determine the assumptions that need to be made and identify the variables in solving the problem.

MEMORY BOX

Simple interest is a reward given to the depositor at a certain rate on the principal amount for a certain period of time. Simple interest can be calculated using the formula $I = Prt$ where I is the interest, P is the principal, r is the interest rate and t is the time in years.

Solution:

(a) In this problem, we know the principal and the interest rate. The interest from the savings is the amount that Jeremy needs besides the RM4 000 to buy the computer. We need to determine how long Jeremy needs to keep his savings in the bank.

(b) **Assumptions:**

- We need to assume that the interest rate does not change during the period of our interest calculation. Otherwise, the formula $I = Prt$ will not be appropriate.
- We also need to assume that the price of the computer does not change when Jeremy has raised the required amount of money.

Variables:

The variables involved are I for interest, P for principal, r for interest rate and t for time in years.

Self Practice 8.1a

1. The journey of a boat moving upstream which covers two places located on the riverbank takes 6 hours. The return journey of the boat moving downstream takes 5 hours. If the speed of the river current is 2 km h^{-1}, what is the speed of the boat on calm water?

(a) Identify and define the problem.

(b) Determine the assumptions that need to be made and identify the variables in solving the problem.

2. In a group, read the question below and find information on the internet.

> You have just completed a degree in marketing and received offers to become a marketing executive by two companies. Company A is in your hometown and offers a lower starting salary and a lower annual salary increment than company B that is in a big city. Which is the better choice?

Explain how you use the mathematical modeling process to solve the above problem.

How to solve real life problems through mathematical modeling?

Learning Standard

Solve real life problems through mathematical modeling which involves the following functions:
(i) Linear
(ii) Quadratic
(iii) Exponential
and communicate the mathematical modeling process implemented.

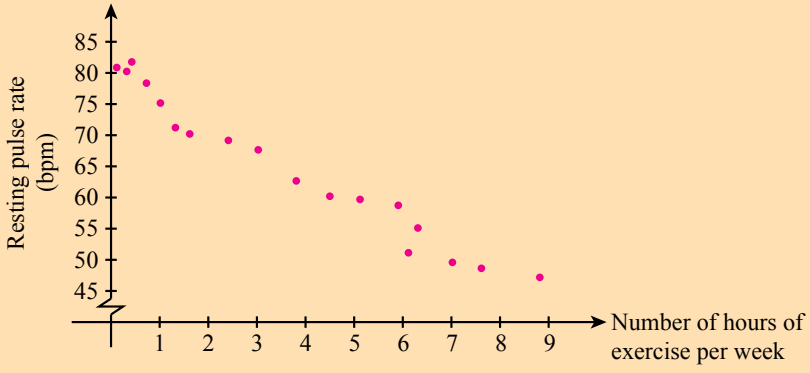
MIND MOBILISATION 1 Pairs

Aim: To apply mathematics in problem solving.

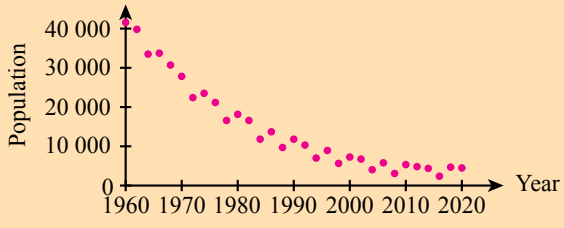
Steps:

1. Based on the information and diagrams given, determine the type of function (linear, quadratic or exponential) that may be used as a model for the data.
2. Discuss your selection.

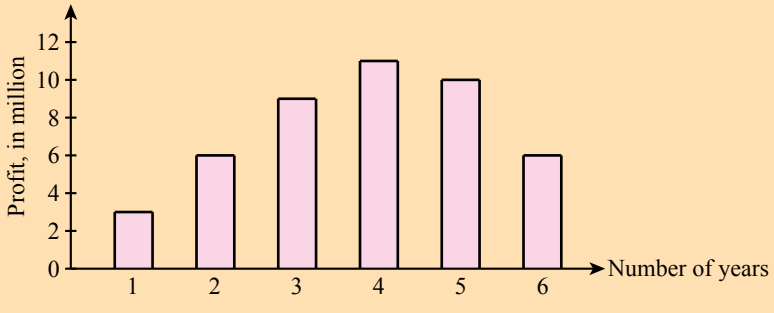
(a) A doctor uses the data to study the resting pulse rate of a normal adult according to the number of hours of exercise per week.



(b) A scientist uses the data to study the population of tigers.



(c) A marketing manager studies the data which shows the relationship between the profit and the number of years a product is in the market.

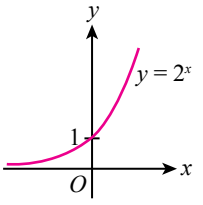


The results of Mind Mobilisation 1 show that we can determine the types of functions that may be used as a model for data by **studying graph patterns**.

Linear function: A graph resembles a straight line.

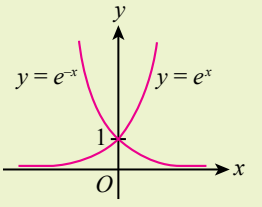
Quadratic function: A curved graph in the form of a parabola. The curve may rise and then fall or fall and then rise.

Exponential function: A curved graph illustrating an increase or decrease in values of data at a sharp rate. The exponential function can be written in the form of equation $y = Ca^x$, where a is a positive real number, $a \neq 1$ and C is an initial value. For example, if $C = 1$ and $a = 2$, graph $y = 2^x$ is shown as the diagram.



Info Bulletin

The diagram below shows the graphs of two exponential functions with base e where $e = 2.7182818284$



Example 2

Amin travelled 405 km on 45 litres of petrol in his car. If Amin wishes to go to a place which is 198 km away by car, how much petrol, in litres, does he need? Solve this problem through mathematical modeling.

Solution:

Identifying and defining the problem

- Determine the volume of petrol required for 198 km.
- We know that the farther we travel, the more petrol we require. Thus, the amount of petrol varies directly with the distance we travelled.

Making assumptions and identifying the variables

- Assume that the driving speed for both 405 km and 198 km routes are the same
- Let x represents the distance travelled and y represents the amount of petrol required
- y varies directly with x, hence $y = kx$ where k is a constant

Applying mathematics to solve problems

Substitute $y = 45$ and $x = 405$ into $y = kx$,

$45 = k(405)$

$$k = \frac{45}{405} = \frac{1}{9}$$

Therefore, $y = \frac{1}{9}x$

This equation describes the relationship between the amount of petrol required and distance travelled.

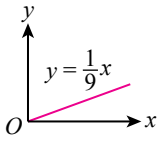
When $x = 198$, $y = \frac{1}{9}(198)$

$= 22$ litres

Hence, 22 litres of petrol is required to travel 198 km.

Verifying and interpreting solutions in the context of the problem

We may not be able to use the linear function model $y = \frac{1}{9}x$ in all situations. For example, if the 405 km route is through towns and cities, and the 198 km route is through highway. Thus, the car will use up petrol at a faster rate in the first route compared to the second route. When this is translated to the real-world situation, the linear function model obtained is not suitable to solve this problem.



Refining the mathematical model

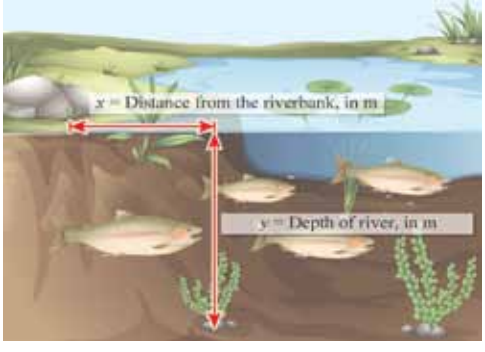
In this problem, we are not able to refine the model due to limited information given.

Reporting the findings

Report the findings of the problem solving based on the interpretation of solutions as shown in the preceding sections.

Example 3

The diagram below shows the cross section of a river. A hydrologist measures the depth of the river, y m, at different distance, x m from the riverbank. The results obtained are given in the following table.



Distance from the riverbank, x m	Depth of the river, y m
0	0
4	1.5
8	2.3
12	2.9
18	2.9
25	1.7
30	0

Show how the hydrologist uses the data above to determine the depth of the river through mathematical modeling.

Solution:

Identifying and defining the problem

How to determine the depth of the river?

Application & Career

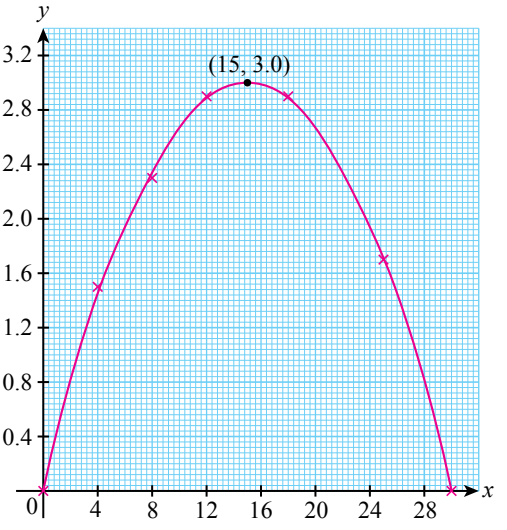
A hydrologist is a scientist who researches the circulation, distribution and quality of water in an environment.

Making assumptions and identifying the variables

- Assume the river is the deepest in the middle with the depth decreasing to 0 at the edges.
- The two variables involved in this study are the depth of the river, y m, and the distance from the riverbank, x m.

Applying mathematics to solve problems

- Write the distance from the riverbank and the depth of the river as a set of ordered pairs (x, y) and draw a graph for the data.
- The data seem to rise and fall in a manner similar to a quadratic function.
- The graph drawn shows the curve of best fit and **resembles the graph of a quadratic function**.
- In mathematical modeling to represent the actual situation, the approximate value is used.
- Based on the graph, the depth of the river is 3 m when the distance from the riverbank is 15 m (approximate).



Verifying and interpreting solutions in the context of the problem

Determine the related quadratic function of the form $y = ax^2 + bx + c$. Determine the constants a, b and c by substituting any three data, for example (0, 0), (25, 1.7) and (30, 0) into the equation.

$$\begin{aligned} 0 &= a(0)^2 + b(0) + c \\ 1.7 &= a(25)^2 + b(25) + c \\ 0 &= a(30)^2 + b(30) + c \end{aligned} \quad \Rightarrow \quad \begin{aligned} 0 &= c \\ 1.7 &= 625a + 25b + c \\ 0 &= 900a + 30b + c \end{aligned}$$

Since $c = 0$, the system of two linear equations in two variables is:

$1.7 = 625a + 25b$ ..... ①

$0 = 900a + 30b$ ..... ②

From ②, $b = -30a$ ..... ③

Substitute ③ into ①, $1.7 = 625a + 25(-30a)$

$1.7 = -125a$

$a = -0.0136$

Substitute $a = -0.0136$ into ③, $b = -30(-0.0136)$

$= 0.408$

Hence, the possible quadratic function is $y = -0.0136x^2 + 0.408x$

Substitute $x = 15$, $y = -0.0136(15)^2 + 0.408(15)$

$= 3.06$ (approximate to the answer obtained from the graph)

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A system of linear equations refer to two or more linear equations involving the same set of variables.

Refining the mathematical model

- For this model, we assume that the river is the deepest in the middle. This may not be true for some other rivers. A new model will be needed if we have new assumptions.
- The answer will be more accurate if more data have been collected.

Reporting the findings

Write a full report following the above modeling framework structure.

Example 4

Compound interest is interest that is calculated based on the original principal and also the accumulated interest from the previous period of savings. At the beginning of a year, Mr Gomez saves RM20 000 in his savings account with an interest rate of 4% per annum and the interest is compounded once a year. Derive a mathematical model for Mr Gomez's total savings after t years of saving. Solve this problem through mathematical modeling.

Solution:

Identifying and defining the problem

- Mr Gomez is given compound interest which is compounded once a year.
- His principal is RM20 000.
- The yearly interest rate is 4%.
- Derive a mathematical model for Mr Gomez's total savings at the end of t years.

Making assumptions and identifying the variables

- Assume Mr Gomez did not withdraw or raise his savings throughout the period of saving.
- The variables are principal amount, RMP, yearly interest rate, r, number of times the interest is compounded, n, and time, t years.

Applying mathematics to solve problems

Keep in mind that when calculating the compound interest, each year we have 100% of the principal, plus 4% of the previous balance. We construct a table as follows.

Year	Principal (RM)	Interest received (RM)	Principal + Interest (RM)	Amount of savings (RM)
1	$20\,000$	$20\,000 \times 0.04$	$20\,000 + 20\,000 \times 0.04$ $= 20\,000(1 + 0.04)$	$20\,000(1.04)$
2	$20\,000(1.04)$	$20\,000(1.04) \times 0.04$	$20\,000(1.04) + 20\,000(1.04) \times 0.04$ $= 20\,000(1.04)(1 + 0.04)$	$20\,000(1.04)^2$
3	$20\,000(1.04)^2$	$20\,000(1.04)^2 \times 0.04$	$20\,000(1.04)^2 + 20\,000(1.04)^2 \times 0.04$ $= 20\,000(1.04)^2(1 + 0.04)$	$20\,000(1.04)^3$
4	$20\,000(1.04)^3$	$20\,000(1.04)^3 \times 0.04$	$20\,000(1.04)^3 + 20\,000(1.04)^3 \times 0.04$ $= 20\,000(1.04)^3(1 + 0.04)$	$20\,000(1.04)^4$
5	$20\,000(1.04)^4$	$20\,000(1.04)^4 \times 0.04$	$20\,000(1.04)^4 + 20\,000(1.04)^4 \times 0.04$ $= 20\,000(1.04)^4(1 + 0.04)$	$20\,000(1.04)^5$

Verifying and interpreting solutions in the context of the problem

Based on the table above, at the end of every year, the amount of savings is a power of 1.04 times the principal, RM20 000, and the power corresponds to the number of years of saving.

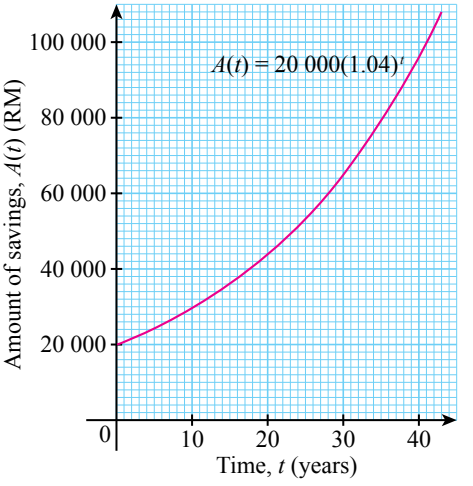
We can generalise this pattern to a mathematical model by letting P represents the principal of the savings, t represents the number of years, and r represents the yearly interest rate. Hence, the mathematical model is

$$A(t) = P(1 + r)^t,$$ ← Note that $1.04 = 1 + 0.04 = 1 + r$

where $A(t)$ is the amount of savings after t years.

This mathematical model is an exponential function that allows us to calculate the amount of savings if the interest is compounded once a year for t years.

The graph on the right shows an exponential function $A(t) = 20\ 000(1.04)^t$. The graph of the amount of savings after t years, $A(t)$, shows an exponential growth as time, t, increases.



Refining the mathematical model

The mathematical model $A(t) = P(1 + r)^t$ is used for annual compounding. In reality, interest is often compounded more frequently, for example, compounded semi-annually or quarterly.

If the interest is compounded monthly, the interest rate r is divided amongst the 12 months because $\frac{1}{12}$ of the rate is applied each month. The variable t in the exponent is multiplied by 12 because the interest is calculated 12 times in a year. Hence, the mathematical model becomes $A(t) = P\left(1 + \frac{r}{12}\right)^{12t}$. In general, if the interest is compounded n times per year, the mathematical model is $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

Reporting the findings

Write a full report following the above modeling framework structure.

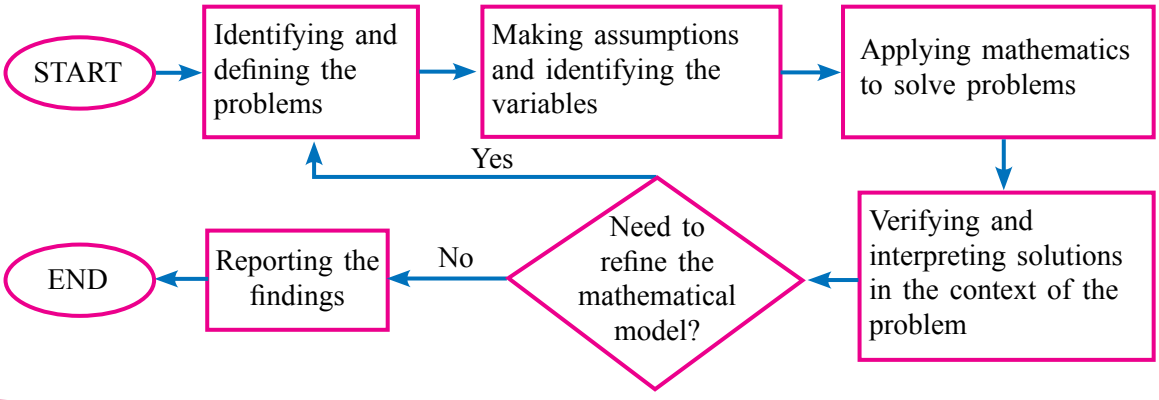
Self Practice 8.1b

1. A runner is training for a marathon, running a total of 32 km per week on a regular basis. The runner plans to increase his running distance $D(x)$, in km, each week by 10%, where x represents the number of weeks of training. Derive a mathematical model for his running distance, $D(x)$. Solve this problem through mathematical modeling.

2. Hyperthermia is a condition where an individual's body temperature is elevated beyond normal. Causes of hyperthermia include dehydration, an excessive loss of body fluids. Why the risk of hyperthermia higher among young children compared to adults in hot weather? Investigate this problem through mathematical modeling.

Summary Arena

MATHEMATICAL MODELING PROCESS



Reflection

At the end of this chapter, I can

explain mathematical modeling.		
solve real life problems through mathematical modeling which involves the following functions: (i) linear (ii) quadratic (iii) exponential and communicate the mathematical modeling process implemented.		

MINI PROJECT

Open the worksheet for this project.

1. Estimate the length of Malaysia's coastline using the 200 km unit and the 100 km unit.
2. Predict the result if the length of coastline is measured using the 50 km unit. Explain.
3. If you continue to decrease the size of the unit of measurement, what do you think will happen to the length of the coastline?
4. Will you ever reach a point where you will find the exact length of the coastline?

Scan the QR code or visit bit.do/MPChap8 for the worksheet.

After you have determined the length of Malaysia's coastline by the method above, find the actual length of Malaysia's coastline from a reliable source. Also find information on how a geographer determine the length of a coastline of a country. Prepare a written report for the mathematical modeling process that you carry out in this project.

Extensive Practice

Scan the QR code or visit bit.do/QuizE08 for interactive quiz

UNDERSTAND

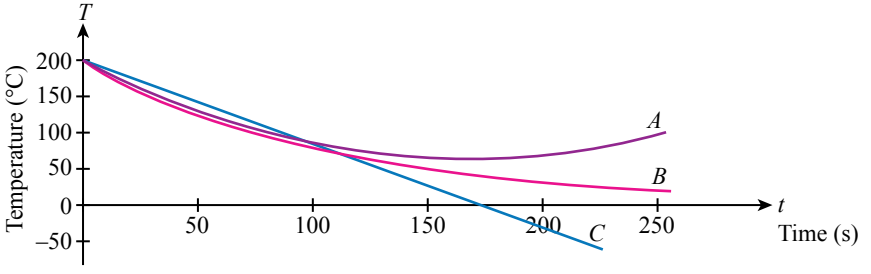
1. State all the components in mathematical modeling.

MASTERY

2. A scientist is studying the cooling patterns of a particular material over time. His research requires heating a sample of the material up to 200°C. Then, he records the temperature of the sample as it is cooled to room temperature. The following table shows the data collected during the first 2 minutes of the cooling process.

Time (s)	0	40	80	120
Temperature (°C)	200	140	100	75

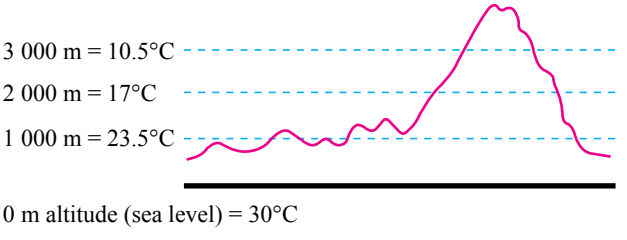
The diagram below shows three possible models for the data, a linear model, a quadratic model and an exponential model.



(a) Which among the graphs of A, B and C, is a linear model, a quadratic model and an exponential model?

(b) Which among the models of A, B and C best describes the temperature of the sample for the range of times $0 \leqslant t \leqslant 250$? Explain why the other models do not fit very well for the range of times given.

3. As altitude increases, the temperature usually decreases. On average, the rate at which the temperature changes with height in the troposphere (the first layer of the Earth's atmosphere) is 6.5°C per 1 000 m. The diagram below shows the cross section of Mount Kinabalu (4 095 m) in Sabah with various temperatures at different altitudes. The temperature at an altitude of 0 m (sea level) is 30°C.

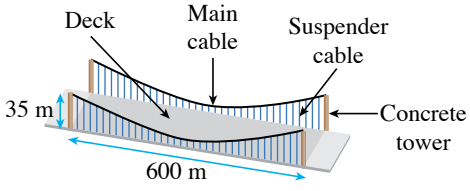


(a) Write an equation that expresses the temperature, T°C as a function of the altitude, x m, for the situation shown in the diagram above.

(b) Verify that your equation in (a) is reasonable by using the data provided in the diagram.
(c) Draw a graph that shows the relationship between x and T. Show a clearly defined scale on each axis and label your axes.
(d) What is a reasonable range for the function $T(x)$? Justify your answer.
(e) What is the gradient of the graph drawn? What is the meaning of the gradient of the graph in terms of temperature and altitude?
(f) What is the y-intercept of the graph drawn? What is the meaning of the y-intercept in terms of temperature and altitude?
(g) Is it reasonable to assume that temperature always decreases at a linear rate as altitude increases? Explain your answer.

CHALLENGE

4. The diagram on the right shows a suspension bridge. The shortest distance from the main cables to the deck is 5 m. Show how an engineer can use the information given to solve the following problems through mathematical modeling.



(a) Determine the function that represents the arc of the main cable.
(b) If the suspender cables are 20 m apart, determine the number of suspender cables needed on both sides of the bridge.
(c) Determine the type of material that should be used to make the suspender cables such that the total cost of the cables is minimum.

Distance between the suspender cables (m)	Type of material	Cost per m (RM)	Total cost of the suspender cables (RM)
15	*A*	750	
20	*B*	1 000	
25	*C*	1 200	

EXPLORING MATHEMATICS

Your class is organising an end of the school year party. You are to order the kind of pizza that suits your classmates most. The things you need to decide on are the pizza outlet, the delivery distance, the cost of the pizza, the size of the pizza, the mode of delivery (self-pick-up, food ordering mobile apps, etc.), the kind of pizza (vegetarian, gluten-free, etc.) and the types of pizza toppings (tomato, cheese, onion, etc.).

Solve this problem through mathematical modeling. You can use variables to represent the changing quantities and use equations to represent the relationship between these quantities. For qualitative types of questions, for example, for the types of pizza toppings, you can create a scoring system that includes calorie values and nutrients. In the process of modeling, you can also make assumptions by restricting the choices to narrow down the scope of your modeling.

CHAPTER 1 Variation

Self Practice 1.1a

1. (a) The value of resistance increases by 10%.
(b) The value of resistance decreases by half.
(c) The temperature decreases by $\frac{1}{4}$.

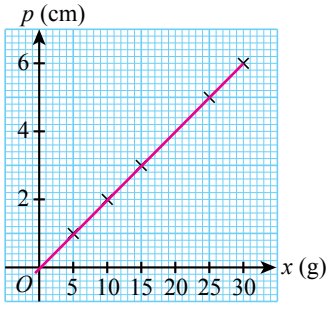
2. (a) The price decreases half of RMx per kilogram.
(b) The price increases two times of RMx per kilogram.

3. (a) The number of bottles of jam is doubled.
(b) The number of bottles of jam is halved.

Self Practice 1.1b

1. (a) y varies directly as x. $y \propto x$
(b) y varies directly as $\sqrt{x}$. $y \propto \sqrt{x}$

2.



Graph of p against x shows a straight line that passes through the origin. Hence, p varies directly as x.

3. (a) $p = 0.5q^3$
(b) $p = 16\sqrt{q}$

4. $x = 8t$

5. (a) 15
(b) 125

6. (a) 1.6
(b) 0.343

7. 15 minutes

8. 6 seconds

9. (a) $x = \frac{1}{12}d$
(b) 3.75 litres

Self Practice 1.1c

1. (a) $s \propto tu$
(b) $v \propto w^2x$
(c) $a \propto b^3\sqrt{c}$
(d) $A \propto rh$

2. (a) $k = 0.5$
(b) $k = 10$

3. (a) $y = \frac{3}{2}z\sqrt{x}$
(b) $y = \frac{3}{8}xz^2$

4. $x = 0.02pr^2$

5. (a) 54
(b) 0.36

6. $x = 16$, $y = 0.3$

7. 16 m s^{-1}

8. 3 300 cm^3

Self Practice 1.1d

1. (a) $V = 0.16t$
(b) 2:15 p.m.

2. (a) $A = 0.5xy$
(b) Area increases by 8%.

3. No. The pizza with diameter of 30 cm is more worth it.

Self Practice 1.2a

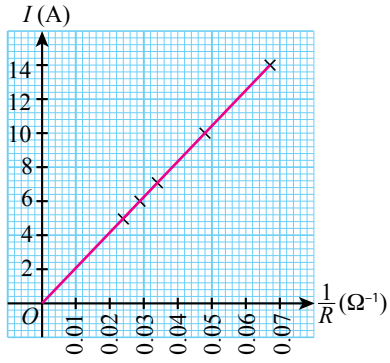
1. (a) The number of days is halved.
(b) The number of days is doubled.

2. (a) The number of gifts decreases halved.
(b) The number of gifts increases by 50%.

Self Practice 1.2b

1. (a) y does not vary inversely as x.
(b) y varies inversely as x^2.
$y \propto \frac{1}{x^2}$

2.



Graph of I against $\frac{1}{R}$ shows a straight line that starts from the origin. Hence, I varies inversely as R.

3. (a) $g = \frac{1.2}{h}$

(b) $g = \frac{9.6}{h^2}$

(c) $g = \frac{0.3}{\sqrt[3]{h}}$

4. (a) 200
(b) 32 000 000
(c) 10
(d) 3.684

5. (a) $s = 4$, $t = 1.2$
(b) $s = 25$, $t = 60$

6. (a) $T = \frac{60}{M}$
(b) $p = 7.5$, $q = 15$

7. $T = \frac{3.17}{\sqrt{g}}$

Self Practice 1.2c

1. (a) R varies inversely as r^2.
(b) 0.24 mm

2. 0.6 cm

3. 12.96 cm

Self Practice 1.3a

1. (a) $w \propto \frac{\sqrt[3]{v}}{x^2}$, $w = \frac{k\sqrt[3]{v}}{x^2}$

(b) $F \propto \frac{GH^3}{\sqrt{t}}$, $F = \frac{kGH^3}{\sqrt{t}}$

(c) $A \propto \frac{s}{t^2}$, $A = \frac{ks}{t^2}$

2. $t = \frac{c}{100p}$

3. (a) $P = 1.44$
(b) $N = 19$

4. $a = 125$; $b = 2.25$

Self Practice 1.3b

1. (a) 400 pieces
(b) The number of tiles needed decreases.

2. (a) 90 km
(b) 70 400 people

Extensive Practice

1. (a) $w \propto x^3$

(b) $a \propto \frac{b}{c^3}$

(c) $p \propto q\sqrt{r}$

(d) $s \propto at^2$

2. $y \propto \frac{1}{\sqrt[3]{x}}$

3. (a) y varies directly as x and z.
(b) e varies inversely as f.
(c) p varies directly as $\sqrt[3]{q}$ and inversely as r.
(d) n varies directly as p and q^2, and inversely as $\sqrt{r}$.

4. (a) Yes (b) No
(c) No (d) Yes

5. (a) 0.5 (b) $\frac{1}{3}$
(c) 1.4

6. $m = \frac{12}{np}$

7. ± 1.4

8. $m = 9.6$, $n = 0.64$

9. (a) $P = \frac{1}{5Q}$
(b) 0.68

10. 0.3 A

11. (a) 154 cm^2
(b) The area of the curved surface will decrease.

12. (a) 0.075
(b) 137.5%

13. Santhami increases his speed by 5.28 km per hour (31.68 km/h – 26.4 km/h).

CHAPTER 2 Matrices

Self Practice 2.1a

1. $[857 \quad 3180 \quad 211]$ or $\begin{bmatrix} 857 \\ 3180 \\ 211 \end{bmatrix}$

2. $\begin{bmatrix} 53 & 52 & 50 \\ 20 & 21 & 20 \\ 47 & 48 & 46 \end{bmatrix}$ or $\begin{bmatrix} 53 & 20 & 47 \\ 52 & 21 & 48 \\ 50 & 20 & 46 \end{bmatrix}$

3. $\begin{bmatrix} 20 & 18 & 15 \\ 12 & 10 & 11 \end{bmatrix}$ or $\begin{bmatrix} 20 & 12 \\ 18 & 10 \\ 15 & 11 \end{bmatrix}$

Self Practice 2.1b

1. (a) 1×2 (b) 2×1
(c) 3×3 (d) 2×3

2. (a) (i) 3×2
(ii) 0
(iii) 9
(b) (i) 3×3
(ii) 16
(iii) 9

3. 2×3, $f_{13} = 2$, $f_{22} = 3$, $f_{11} = -8$

4. 4

Self Practice 2.1c

1. (a) Equal
(b) Equal
(c) Not equal
(d) Not equal

2. (a) $x = 6$, $y = 2$, $z = \frac{1}{2}$
(b) $x = 2$, $y = -\frac{7}{2}$, $z = -4$

Self Practice 2.2a

1. (a) Yes (b) No
(c) No (d) Yes

2. (a) $\begin{bmatrix} 10 & 6 \\ 4 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 14 & -4 \\ -10 & 10 \end{bmatrix}$

3. (a) $\begin{bmatrix} 13 & 9 & 10 \\ -2 & 8 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -8 \\ -6 & 13 \end{bmatrix}$

(c) $\begin{bmatrix} -16 \\ 18 \end{bmatrix}$ (d) $\begin{bmatrix} 16 & 14 \\ 0 & 20 \end{bmatrix}$

4. $a = 3$, $b = -17$

5. $x = 3$, $y = 1$, $z = 0$

6. $\begin{bmatrix} -15 & 2 \\ 11 & 1 \\ -2 & 6 \end{bmatrix}$

7. $\begin{bmatrix} 2650 \\ 1560 \end{bmatrix} + \begin{bmatrix} 1890 \\ 910 \end{bmatrix} - \begin{bmatrix} 930 \\ 540 \end{bmatrix} - \begin{bmatrix} 850 \\ 260 \end{bmatrix}$
$= \begin{bmatrix} 2760 \\ 1670 \end{bmatrix}$

Self Practice 2.2b

1. (a) $\begin{bmatrix} -21 \\ 6 \end{bmatrix}$ (b) $[6.6 \quad 3]$

(c) $\begin{bmatrix} 3 & -5 \\ -\frac{3}{2} & 4 \\ \frac{9}{4} & \frac{1}{4} \end{bmatrix}$ (d) $\begin{bmatrix} -0.8 & -16 \\ 18 & -5 \end{bmatrix}$

(e) $\begin{bmatrix} 12 & -1.2 & 13.2 \\ 3.6 & 8.4 & -6 \end{bmatrix}$ (f) $\begin{bmatrix} -5 \\ \frac{9}{2} \\ 1 \end{bmatrix}$

2. (a) $\begin{bmatrix} 7 & 14 \\ 16.5 & -2 \\ 2 & -19 \end{bmatrix}$ (b) $\begin{bmatrix} -12 \\ 23 \end{bmatrix}$

(c) $[14 \quad -16 \quad 10]$ (d) $\begin{bmatrix} 5 & -1 \\ -2.2 & 2.1 \end{bmatrix}$

3. $(E + F) + G = \begin{bmatrix} 2 & 28 \\ 5 & 15 \end{bmatrix} + \begin{bmatrix} -1 & 10 \\ -8 & 5 \end{bmatrix}$
$= \begin{bmatrix} 1 & 38 \\ -3 & 20 \end{bmatrix}$

$E + (F + G) = \begin{bmatrix} 9 & 6 \\ 2 & 11 \end{bmatrix} + \begin{bmatrix} -8 & 32 \\ -5 & 9 \end{bmatrix}$
$= \begin{bmatrix} 1 & 38 \\ -3 & 20 \end{bmatrix}$

4. $\begin{bmatrix} -5.1 \\ 6.3 \end{bmatrix}$

5. $a = 20$, $b = \frac{1}{4}$, $c = 1$

6. $\begin{bmatrix} -\frac{3}{2} & 3 \end{bmatrix}$

7. $\begin{bmatrix} 85 & 70 \\ 110 & 98 \end{bmatrix} - \begin{bmatrix} 33 & 24 \\ 42 & 40 \end{bmatrix} = \begin{bmatrix} 52 & 46 \\ 68 & 58 \end{bmatrix}$

Self Practice 2.2c

1. (a) Yes, 2×1
(b) Yes, 2×3
(c) No
(d) No
(e) Yes, 2×3
(f) No

2. (a) $\begin{bmatrix} -5 & 19 \\ -7 & 16 \end{bmatrix}$ (b) $\begin{bmatrix} -12 & 16 \\ 9 & -23 \\ 8 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} -8 \\ 28 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} -10 \\ -26 \end{bmatrix}$

(e) $\begin{bmatrix} 7 & -2 \\ -6 & 19 \end{bmatrix}$ (f) $\begin{bmatrix} 8 & 15 \\ 45 & -82 \end{bmatrix}$

3. $x = 5, y = 2$

4. $r = -3.5, s = 2$

5. (a) $p = 2, q = 7, r = 1$
(b) $p = -1, q = 3, r = 6$
(c) $p = 3, q = -23.2, r = 5$
(d) $p = -7, q = -15, r = 15$

6. $\begin{bmatrix} 40 & 28 & 36 \\ 42 & 36 & 30 \\ 35 & 25 & 42 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 640 \\ 678 \\ 627 \end{bmatrix}$

Self Practice 2.2d

1. (a) No. It is not a square matrix.
(b) No. The elements along the main diagonal are not 1.
(c) No. The elements along the main diagonal are not 1.
(d) Yes.
(e) Yes.
(f) No. The elements along the main diagonal are not 1.

2. $CD = DC = C$. Hence, D is an identity matrix.

3. (a) $\begin{bmatrix} 10 & 3 \\ 1 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 11 & 15 \\ 3 & 18 \end{bmatrix}$

(c) $\begin{bmatrix} 11 & 4 \\ -20 & 15 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 2 \\ 6 & 2 \end{bmatrix}$

Self Practice 2.2e

1. (a) Not inverse matrices
(b) Not inverse matrices
(c) Not inverse matrices
(d) Yes. Inverse matrices

Self Practice 2.2f

1. (a) Exists, $\begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{bmatrix}$

(b) Exists, $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

(c) Exists, $\begin{bmatrix} -3 & -\frac{5}{3} \\ -1 & -\frac{2}{3} \end{bmatrix}$

(d) Does not exist

2. (a) $\begin{bmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$ (d) $\begin{bmatrix} -\frac{7}{4} & -\frac{5}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

3. (a) $\frac{3}{2}$ (b) 4

4. $\begin{bmatrix} -1 & 10 \\ \frac{1}{2} & -4 \end{bmatrix}$

Self Practice 2.2g

1. (a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -14 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ -10 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 2 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 1 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ -12 \end{bmatrix}$

(g) $\begin{bmatrix} 2 & -5 \\ \frac{1}{5} & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & -4 \\ 0.8 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 100 \\ -14 \end{bmatrix}$

2. (a) $x = 5, y = 0$
(b) $x = -2, y = -1$
(c) $x = 3, y = -2$
(d) $x = 2, y = -1$

(e) $x = 0.5, y = 3$
(f) $x = 1.5, y = 0.3$
(g) $p = -2, q = 2$
(h) $m = 3, n = 2$

Self Practice 2.2h

1. the price for a curry puff with sardine filling = RM0.60,
   the price for a curry puff with potato filling = RM0.80
2. the time spent for swimming = 6 hours,
   the time spent for playing badminton = 4 hours
3. the price for one kilogram of papaya = RM4,
   the price for one kilogram of banana = RM5
4. the number of cars = 29,
   the number of motorcycles = 37
5. the dividend rate of Unit Trust P = 4%,
   the dividend rate of Unit Trust Q = 5%

Extensive Practice

1. 3 rows and 2 columns
2. 3×1
3. -6
4. $E + E + E = \begin{bmatrix} -1 & 2 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 5 & 4 \end{bmatrix}$
   $= \begin{bmatrix} -3 & 6 \\ 15 & 12 \end{bmatrix}$
   $3E = 3\begin{bmatrix} -1 & 2 \\ 5 & 4 \end{bmatrix}$
   $= \begin{bmatrix} -3 & 6 \\ 15 & 12 \end{bmatrix}$
5. $\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
6. $r = -3, s = 3$
7. $\begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$
8. $x = 1, y = -7, z = 9$
9. (a) $\begin{bmatrix} -37 & 54 \\ 81 & -118 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & 6 \\ 9 & -14 \end{bmatrix}$
10. $p = 2, q = 6$
11. (a) $\begin{bmatrix} -1 & 2 \\ 3 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -19 \end{bmatrix}$
    (b) $x = -1, y = 2$
12. the number of male participants = 80,
    the number of female participants = 48
13. $p = -2q$
14. (a) $\begin{bmatrix} 80 & 70 \\ 60 & 75 \\ 74 & 84 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$
    (b) 76
    (c) Computer Science
15. (a) $6p + 4q = 16$
    $95p + 110q = 300$
    (b) the number of glasses of drink P = 2,
    the number of glasses of drink Q = 1
16. the number of air conditioner K = 32,
    the number of air conditioner L = 18

CHAPTER 3 Consumer Mathematics: Insurance

Self Practice 3.1a

1. (a) Insurance company: *Syarikat Insurans Bersatu*
   Policyholder: Encik Daud
   (b) RM300 000
   (c) RM100
   (d) Personal accident
2. Costs of injury treatment, loss and car damage incurred by the third party only.

Self Practice 3.1b

1. The risk of being involved in an accident for occupation in class 3 is higher than class 1 and class 2.
2. (a) Smoker: RM330.00; Non-smoker: RM254.40
   (b) Smoker: RM175.20; Non-smoker: RM147.60
3. Comprehensive policy = RM1 789.62; Third-party, fire and theft policy = RM1 342.21; third-party policy = RM117.18

Extensive Practice

1. (i) The risk of burglary in the premise.
   (ii) The risk of goods damaged during transportation.
   (iii) The risk of the premise catching fire.
   (Accept other suitable answers)

2. To restore the financial position of the policyholder to the pre-loss condition. Thus, the policyholder cannot earn profit from the compensation of the loss.

3. Personal accident insurance provides coverage to the policyholder in the occurrence of injury, disability or death resulting directly from accident. Life insurance pays compensation upon the death of the policyholder due to natural death, illness or accident within the coverage period whereas the medical and health insurance covers medical expenses, critical illness benefit and more.

4. Third-party, fire and theft policy or comprehensive policy.

5. No. Vanitha does not have the insurable interest upon the car because her friend is still the holder of the insurance policy of the car.

6. (a) RM50.38
   (b) (i) RM796
   (ii) RM244

7. He cannot claim for the loss of RM90. For the loss of RM240, the compensation payment is RM90 and for the loss of RM300, the compensation payment is RM150.

8. 2019: Amount borne by Madam Karen = RM5 000
   Compensation payment = RM125 000
   2020: Amount borne by Madam Karen = RM5 000
   Compensation payment = RM7 000

9. (a) RM910 000
   (b) (i) RM910 000
   (ii) RM546 000

10. (a) RM200 000
    (b) No. Encik Adam will receive the maximum compensation payment of RM200 000, which is the face value of the insurance he purchased.

11. Cost borne by Sofia = RM2 400
    Compensation payment = RM8 400

CHAPTER 4 Consumer Mathematics: Taxation

Self Practice 4.1a

1. As a source of government revenue, as a government policy implementation tool, as a control of sales of goods or services, as a financial tool to stabilise the economy.

2. Real Property Gains Tax serves as a government policy implementation tool. This tax is imposed to reduce excessive property investment activities.

Self Practice 4.1b

1. Property assessment tax is a tax levied on all holdings or properties in order to cover municipal expenses.

2. Mr Tan can be fined not exceeding RM2 000.

3. Manufacturers or importers with sales value of taxable goods exceeding RM500 000 per annum shall register under Sales Tax Act 2018.

4.

Quit rent	Property assessment tax
collected by state land authority, which is State Land Office	collected by local authority, such as municipal council or district council

5. Someone who deliberately evades income tax can be fined RM1 000 up to RM20 000 or imprisonment of not exceeding 3 years or both and penalty of 300% of the amount of tax which had been undercharged.

Self Practice 4.1c

1. (a) RM75 840
   (b) RM52 410
   (c) RM49 960
   (d) RM95 021

2. (a) RM12 971.20
   (b) RM3.20
   (c) RM121.60
   (d) RM885.16

3. RM881.84
4. RM2 115.31
5. Mr Subra had to pay RM712 and his wife had to pay RM68. Hence, the total income tax for Mr Subra and his wife was RM780.
6. Road tax for motorcycle 220 cc = RM50
   Road tax for car 1 860 cc = RM310
7. RM900
8. RM45
9. RM21.20

Self Practice 4.1d

1. RM2 399.34
2. (a) RM99 950
   (b) RM8 389.50
   (c) Encik Jamil should not make any additional tax payment to IRB. The total amount of *PCB* (RM16 800) is more than his income tax (RM8 389.50).

Extensive Practice

1. Tax can be levied in line with government policy, so that the policy is more effective. Real Property Gains Tax is imposed when the government wants to reduce excessive property investment activities. Digital Service Tax is imposed to create a balanced competing platform for local companies. Departure Levy is imposed to an individual who departs from Malaysia by air transportation to encourage domestic tourism development.
2. (a) Inland Revenue Board
   (b) Local authority, such as municipal council or district council
   (c) Road Transport Department
   (d) Royal Malaysian Customs Department
3. Income tax
4. Local authority could seize the portable properties of the owner who does not pay property assessment tax.
5. Tax reliefs are items or expenses, which are not taxed, for the benefit of the individual or family members.
6. Tax rebate of RM400 will be given to taxpayer if the chargeable income is not exceeding RM35 000. The total payment of *zakat* or *fitrah* is also a rebate.
7. RM61.37
8. RM63 500
9. RM0
10. 1 800 cc
11. 4.5%
12. (a) The stall might not generate revenue exceeding RM1 500 000 per annum.
    (b) RM13.96
13. RM3 508
14. (a) RM31 970
    (b) Yes. Eng Huat is eligible for tax rebate of RM400 because his chargeable income is less than RM35 000.
    (c) RM109.10
15. (a) RM60 300
    (b) RM2 642
    (c) Puan Aida should not make any additional income tax payment because the total amount of *PCB* is more than her income tax.

CHAPTER 5 Congruency, Enlargement and Combined Transformations

Self Practice 5.1a

1. (a) congruent
   (b) not congruent
   (c) congruent
   (d) not congruent
2. *A* and *F* are congruent,
   B and *E* are congruent

Self Practice 5.1b

1. (a) $PR = XZ$ and $\angle QPR = \angle YXZ$
   or $QR = YZ$ and $\angle PQR = \angle XYZ$
   (b) The area of triangles *PQR* and *XYZ* must be equal.
   $PR = XZ$ and $\angle PRQ = \angle XZY$
   or $PR = XZ$ and $\angle PQR = \angle XYZ$
   or $QR = YZ$ and $\angle QPR = \angle YXZ$
   or $QR = YZ$ and $\angle PRQ = \angle XZY$
2. (a) Side-Side-Side (SSS)
   (b) Angle-Side-Angle (ASA)

Self Practice 5.1c

1. $AC = CD$, $\angle ABC = \angle DEC$, $\angle BAC = \angle EDC$
Satisfies the Angle-Angle-Side (AAS) rule.
Hence, ΔABC and ΔDEC are congruent.
or
$AC = CD$, $\angle BAC = \angle EDC$, $\angle BCA = \angle ECD$
Satisfies the Angle-Side-Angle (ASA) rule.
Hence, ΔABC and ΔDEC are congruent.

2. (a) 12 cm
(b) 97°

3. (a) 212 m
(b) 1620 m^2

Self Practice 5.2a

1. (a) similar
(b) not similar
(c) not similar
(d) similar

2. A and I are similar,
B and G are similar,
D and F are similar

Self Practice 5.2b

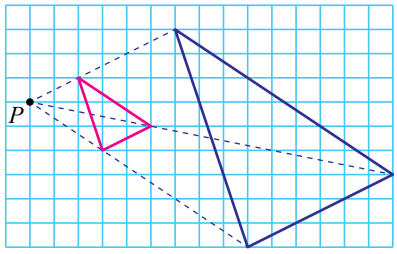
1. (a) Enlargement at centre (–4, 4) with scale factor 3
(b) Enlargement at centre (6, –5) with scale factor 2
(c) Enlargement at centre (5, 2) with scale factor $\frac{1}{2}$
(d) Enlargement at centre (7, 4) with scale factor $-\frac{3}{2}$

2. (a) Enlargement at centre P with scale factor $\frac{5}{2}$
(b) Enlargement at centre P with scale factor $\frac{8}{5}$
(c) Enlargement at centre P with scale factor $\frac{1}{4}$

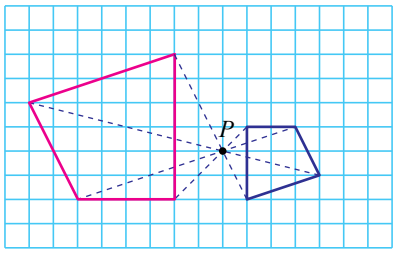
Self Practice 5.2c

1. (a) A = object, B = image
(b) M = image, N = object

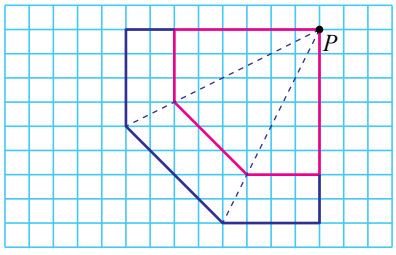
2. (a)



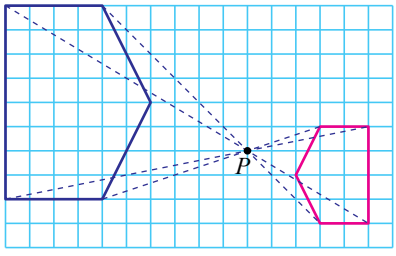
(b)



3. (a)



(b)



Self Practice 5.2d

1.

Area of the object	Area of the image	Scale factor, k
18 unit^2	72 unit^2	2 or –2
54 m^2	6 m^2	$\frac{1}{3}$
20 cm^2	31.25 cm^2	$\frac{5}{4}$

Self Practice 5.2e

1. (a) $-\frac{2}{5}$ (b) 87 cm^2
2. 40 m
3. 51 cm^2
4. 50

Self Practice 5.3a

1. (a) Pentagon II (b) Pentagon III

2. (a) Triangle P (b) Triangle S

3. (a) (i) Point E (ii) Point G
(b) Point D

Self Practice 5.3b

1. (a) combined transformation **PQ** satisfies commutative law
(b) combined transformation **PR** does not satisfy commutative law

Self Practice 5.3c

1. Transformation **A** is a rotation of 90° anticlockwise at centre (4, 6).

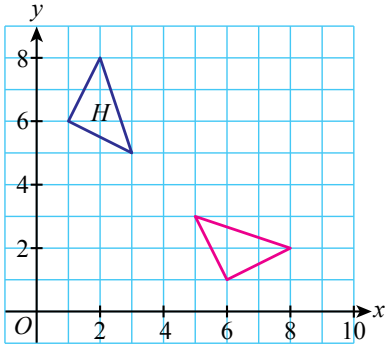
Transformation **B** is a reflection on line $y = 4$.

2. Transformation **P** is an enlargement at centre (7, 2) with scale factor 2.

Transformation **Q** is a translation $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

3. (a) Transformation **B** is a reflection on line $y = 3$
(b) rotation of 180° at centre (5, 3)

4. (a)



(b) reflection on line $y = x$

Self Practice 5.3d

1. (a) Transformation **P** is an enlargement at centre (5, 4) with scale factor 2.

Transformation **Q** is a translation $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

(b) enlargement at centre (1, 10) with scale factor 2
(c) 2.5 unit^2

2. (a) $y = -x$
(b) enlargement at centre (–5, –4) with scale factor $\frac{5}{2}$
(c) 315 m^2

Self Practice 5.4a

1. (a) Yes
(b) Yes
(c) No
(d) No

Self Practice 5.4b

1. (a) reflection on line PQ or translation $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$ or rotation of 120° clockwise at centre P
(b) reflection on line SR or rotation of 120° anticlockwise at centre S
(Accept other correct answers)

Extensive Practice

1. 112°

2. 10.5 cm

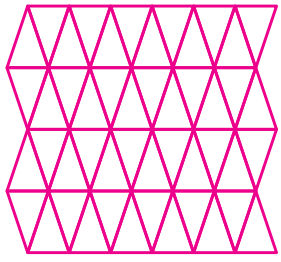
3. enlargement at centre (3, 3) with scale factor 3

4. $\frac{4}{3}$ or $-\frac{4}{3}$

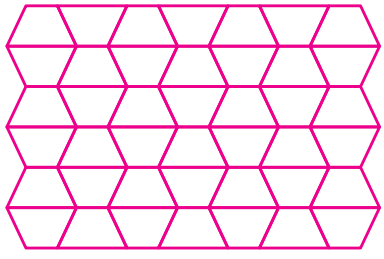
5. (8, 8)

6. rotation of 90° anticlockwise at centre (3, 6)

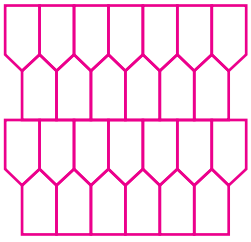
7. (a)



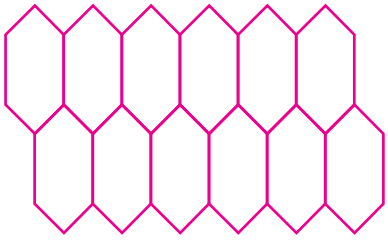
(b)



(c)



(d)



8. (a) (i) enlargement at centre (6, 3) with scale factor $\frac{2}{3}$

(ii) reflection on line $x = 5$

or

(a) (i) reflection on line $x = 5$

(ii) enlargement at centre (4, 3) with scale factor $\frac{2}{3}$

(b) 22.5 unit^2

9. $\frac{5}{2}$

10. 1 215 cm

11. (a) 1 003.96 m

(b) 20 000 m^2

12. Transformation **X** is a reflection on line *JM*.

Transformation **Y** is a rotation of 36° anticlockwise at centre *J*.

CHAPTER 6 Ratios and Graphs of Trigonometric Functions

Self Practice 6.1a

1. (a) 83° (b) 9°
(c) 82° (d) 15.8°
(e) 73.6° (f) 54.3°
(g) 51°7′ (h) 35°42′

2. (a) sin 101° = sin 79°
(b) cos 194° = –cos 14°
(c) tan 246° = tan 66°
(d) tan 294.5° = –tan 65.5°
(e) sin 339.8° = –sin 20.2°
(f) cos 112.3° = –cos 67.7°
(g) cos 287°45′ = cos 72°15′
(h) tan 96°31′ = –tan 83°29′
(i) sin 203°26′ = –sin 23°26′

3. sin $\theta = -0.8290$, cos $\theta = 0.5592$, tan $\theta = -1.4825$

Self Practice 6.1b

1. (a) 0.7880 (b) –0.8290
(c) –0.3907 (d) –0.9613
(e) 0.3256 (f) –0.6018
(g) 0.8693 (h) –1.0355
(i) –0.9657 (j) –0.7983
(k) –0.3140 (l) –0.1616
(m) 0.6108 (n) –4.4615
(o) –0.3040

Self Practice 6.1c

1. (a) sin 120° = $\frac{\sqrt{3}}{2}$, cos 120° = $-\frac{1}{2}$, tan 120° = $-\sqrt{3}$

(b) sin 135° = $\frac{1}{\sqrt{2}}$, cos 135° = $-\frac{1}{\sqrt{2}}$, tan 135° = –1

(c) sin 210° = $-\frac{1}{2}$, cos 210° = $-\frac{\sqrt{3}}{2}$, tan 210° = $\frac{1}{\sqrt{3}}$

(d) sin 240° = $-\frac{\sqrt{3}}{2}$, cos 240° = $-\frac{1}{2}$, tan 240° = $\sqrt{3}$

(e) sin 315° = $-\frac{1}{\sqrt{2}}$, cos 315° = $\frac{1}{\sqrt{2}}$, tan 315° = –1

(f) sin 330° = $-\frac{1}{2}$, cos 330° = $\frac{\sqrt{3}}{2}$, tan 330° = $-\frac{1}{\sqrt{3}}$

2. Angle θ is in quadrant I:

cos $\theta = \frac{1}{2}$, tan $\theta = \sqrt{3}$

Angle θ is in quadrant II:

cos $\theta = -\frac{1}{2}$, tan $\theta = -\sqrt{3}$

Self Practice 6.1d

1. (a) 70° or 110°
(b) 21° or 339°
(c) 40° or 220°
(d) 129.9° or 309.9°
(e) 127° or 233°
(f) 227° or 313°
(g) 125.7° or 234.3°
(h) 237.1° or 302.9°
(i) 36.5° or 216.5°

2. 250.1°

Self Practice 6.1e

1. (a) $-\frac{4}{5}$ (b) $\frac{3}{5}$

2. (a) Quadrant I and II
(b) 54° or 126°

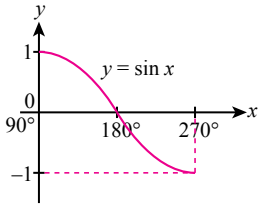
3. (a) 122.08°
(b) 9.68 m
(c) –0.5534

4. (a) 150° (b) $-\frac{\sqrt{3}}{2}$

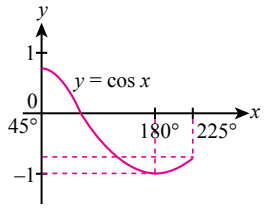
Self Practice 6.2a

1. (a) cosine
(b) tangent
(c) sine

2. (a)



(b)



3. Graph I: $y = \tan x$; no maximum value, minimum value = 0
Graph II: $y = \sin x$; maximum value = 1, minimum value = 0
Graph III: $y = \cos x$; maximum value = 1, minimum value = 0

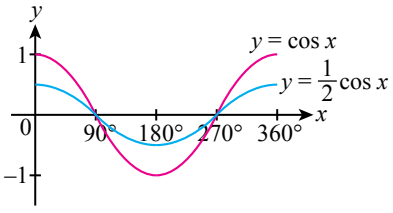
4. (a) $x = 90°$ when $y = 1$
(b) $x = 0°, 360°$ when $y = 1$
(c) No maximum value

5. (a) cosine, $m = 270°$
(b) sine, $m = 90°$

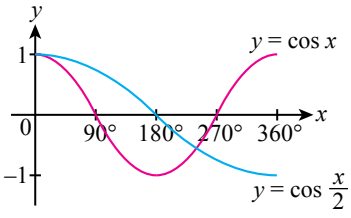
Self Practice 6.2b

1. (a) 4, 360°
(b) 3, 180°
(c) 2, 120°
(d) 1, 90°
(e) 4, 180°
(f) 3, 120°
(g) No, 60°
(h) No, 540°
(i) No, 90°

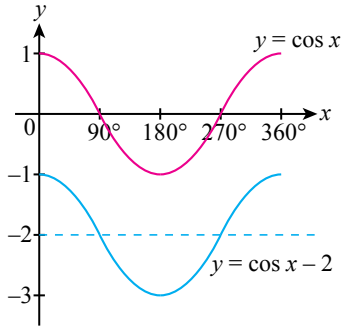
2. (a)



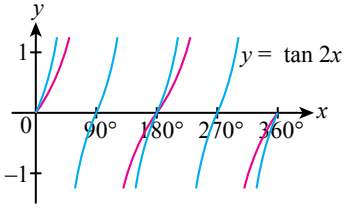
(b)



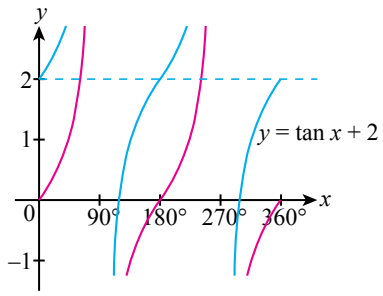
(c)



3. (a)



(b)



Self Practice 6.2c

1. (a) $y = 3 \cos 30x + 9$
(b) 0, 12, 24

2. (a) $s = 500 \tan x$
(b) No amplitude, period = 180°

3. $y = 1.1 \sin 15x + 2.9$

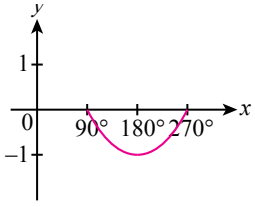
Extensive Practice

1. (a) $\tan 154° = -\tan 26°$
(b) $\sin 234° = -\sin 54°$
(c) $\cos 314° = \cos 46°$

2. (a) –0.4384 (b) 0.6160
(c) –0.8131

3. 144° or 324°

4.

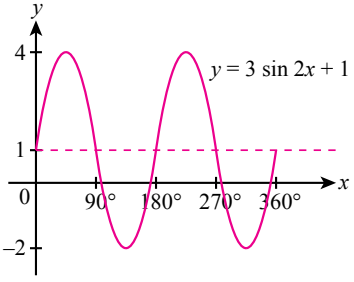


5. maximum value = 2, minimum value = –4

6. (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{3}}$

7. 0.6613

8.



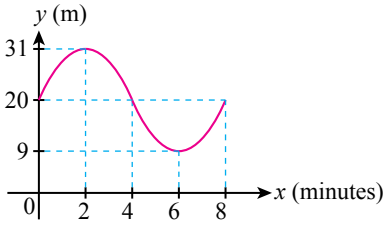
9. (a) $y = \sin x$ (b) 30°

10. (a) cosine (b) 3 ampere
(c) 1.6 seconds

11. $A\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $B\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $C\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

12. 11.5 m

13. (a) sine
(b)



$y = 11 \sin 45x + 20$

14. $s = 5 \tan x$

CHAPTER 7 Measures of Dispersion for Grouped Data

Self Practice 7.1a

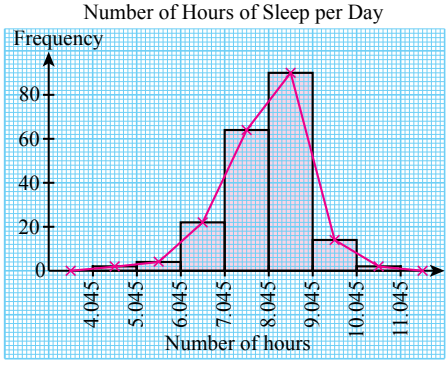
1.

Class	Frequency	Lower limit	Upper limit	Midpoint	Lower boundary	Upper boundary
0 – 9	7	0	9	4.5	0	9.5
10 – 19	21	10	19	14.5	9.5	19.5
20 – 29	10	20	29	24.5	19.5	29.5
30 – 39	10	30	39	34.5	29.5	39.5
40 – 49	2	40	49	44.5	39.5	49.5

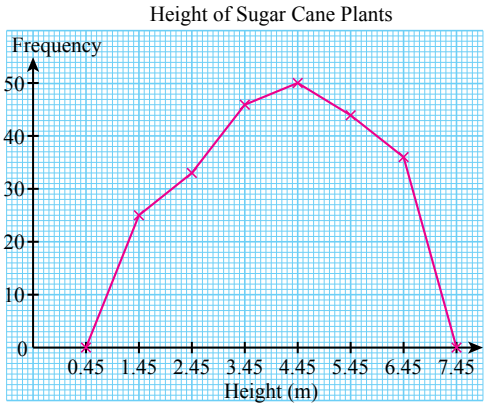
2.

Midpoint	Lower limit	Upper limit	Lower boundary	Upper boundary	Cumulative frequency
2.2	2.0	2.4	1.95	2.45	9
2.7	2.5	2.9	2.45	2.95	24
3.2	3.0	3.4	2.95	3.45	48
3.7	3.5	3.9	3.45	3.95	68
4.2	4.0	4.4	3.95	4.45	78

3.



4.



Self Practice 7.1b

1. (a) The distribution shape of the marks of Arif group is almost uniform whereas the marks of Bestari group is bell-shaped with extreme values.

(b) The marks of Arif group are more dispersed than the Bestari group.

(c) Bestari group. Most of the marks are better.

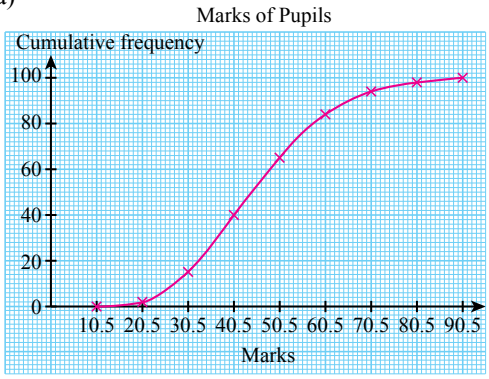
2. (a) The distribution shape in location A is skewed to the right, whereas the distribution shape in location B is bell-shaped.

(b) The car speed in location A is less dispersed compared to the car speed in location B.

(c) Location B is a highway and location A is a housing area.

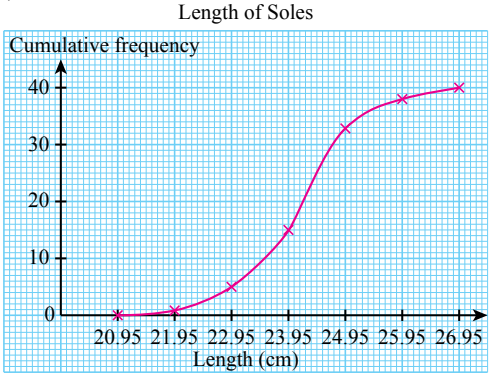
Self Practice 7.1c

1. (a)



(b) (i) first quartile = 35
(ii) median = 44.5
(iii) third quartile = 55

2. (a)

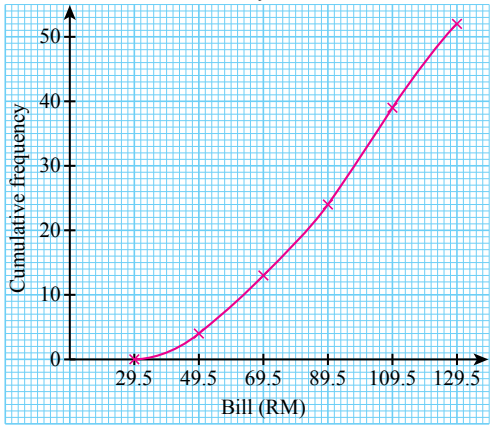


(b) (i) 23.35 (ii) 24.3
(iii) 25.1

(c) 70%

Self Practice 7.2a

1.



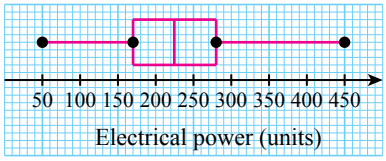
Range = 80 means the difference between the highest electricity bill and the lowest electricity bill is RM80.
Interquartile range = 40 means the difference between the highest electricity bill and the lowest electricity bill that lies in the middle 50% of the distribution is RM40.

2. (a) Variance = 9.50; standard deviation = 3.08

(b) Variance = 266.91; standard deviation = 16.34

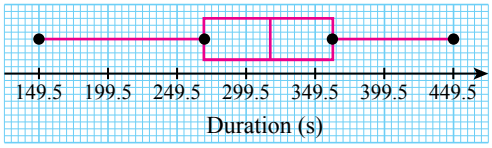
Self Practice 7.2b

1. (a)



(b) The data is slightly skewed to the right.

2. (a)



(b) The data is slightly skewed to the left.

Self Practice 7.2c

1. Machine P, mean = 10.51, standard deviation = 1.33
Machine Q, mean = 11.65, standard deviation = 1.04
Machine Q shows better performance in terms of air pressure accuracy because the mean lies within the range and the smaller standard deviation implies that it is more consistent.

2. Brand X, mean = 2.78, standard deviation = 1.12
Brand Y, mean = 2.1, standard deviation = 1.19
Brand X battery. The larger mean implies that the lifetime is longer and the smaller standard deviation implies that the lifetime is more consistent.

Self Practice 7.2d

1. (a) No. The distribution is not symmetrical as most of the blood glucose readings are concentrated on the right side of the histogram, which are higher.
(b) Mean = 6.82, standard deviation = 1.35
(c) The standard deviation of the blood glucose readings between 6.0 mmol/L and 8.9 mmol/L are lower because the dispersion of the data is smaller.

2. (a) The mean and range of the price of the rice in supermarkets P and Q are the same, which are mean 32 and range 14.
(b) The price distribution of the rice in supermarket P is more symmetrical because the values of the mean and median are the same, and the median lies in the middle of the interquartile range.
(c) The interquartile range of supermarket P is higher than supermarket Q with its median lying in the middle whereas, the interquartile range for supermarket Q is more concentrated where the median and third quartile are the same.

Extensive Practice

1. (a)

Lower limit	Upper limit	Midpoint	Lower boundary	Upper boundary
10	14	12	9.5	14.5
15	19	17	14.5	19.5
20	24	22	19.5	24.5
25	29	27	24.5	29.5
30	34	32	29.5	34.5

(b)

Lower limit	Upper limit	Midpoint	Lower boundary	Upper boundary
25.0	25.9	25.45	24.95	25.95
26.0	26.9	26.45	25.95	26.95
27.0	27.9	27.45	26.95	27.95
28.0	28.9	28.45	27.95	28.95
29.0	29.9	29.45	28.95	29.95

(c)

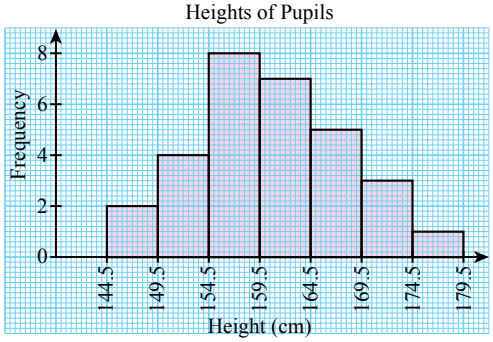
Lower limit	Upper limit	Midpoint	Lower boundary	Upper boundary
0	0.24	0.12	0	0.245
0.25	0.49	0.37	0.245	0.495
0.50	0.74	0.62	0.495	0.745
0.75	0.99	0.87	0.745	0.995

2. (a) Ogive (b) Histogram
(c) Frequency polygon

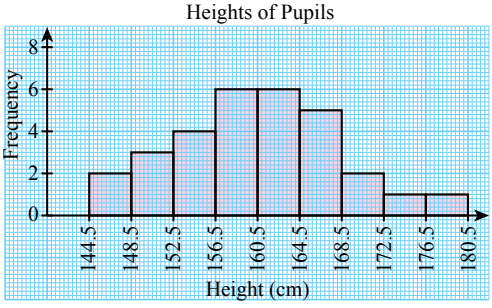
3. (a) C (b) D
(c) A (d) B

4. (a)

Height (cm)	Lower boundary	Upper boundary	Tally	Frequency
145 – 149	144.5	149.5	//	2
150 – 154	149.5	154.5	////	4
155 – 159	154.5	159.5	## ///	8
160 – 164	159.5	164.5	## //	7
165 – 169	164.5	169.5	##	5
170 – 174	169.5	174.5	///	3
175 – 179	174.5	179.5	/	1

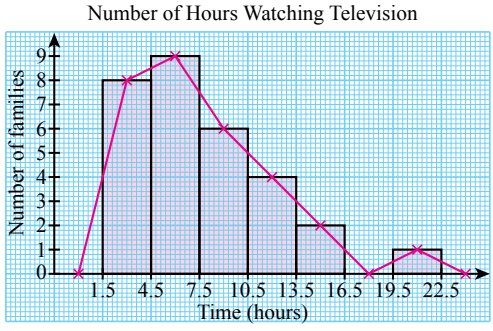


(b)



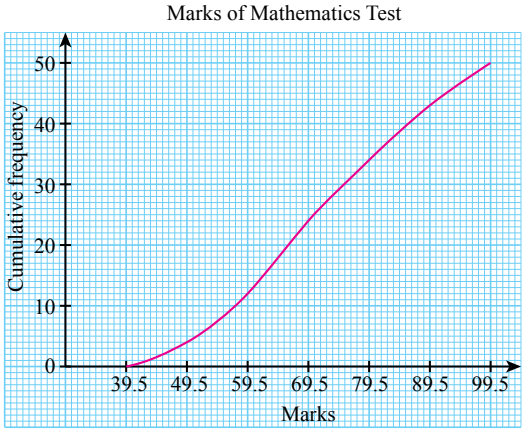
(c) Histogram (a) is slightly skewed to the right where the maximum number of pupils is in the range of 155 to 159 cm. Histogram (b) is almost symmetrical and has two modal heights that ranges between 157 to 164 cm. This shows that the shape of the distribution highly depends on the size of the class interval of grouped data used.

5. (a)



(b) The distribution of the data is skewed to the right where most of the families watch television less than 7 hours per week.

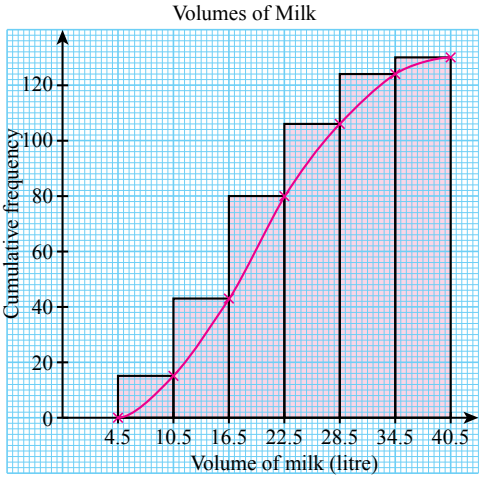
6.



(a) 50 (b) 23

(c) $P_{40} = 66.5$, $P_{80} = 86$

7. (a), (b)



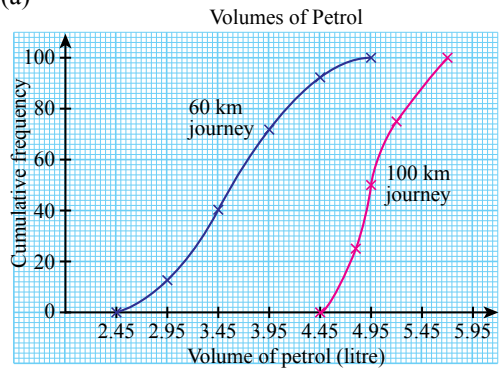
Estimated interquartile range = 11.7 litres

8. Before taking the medicine, mean = 151.6, standard deviation = 15.23
After taking the medicine, mean = 142, standard deviation = 13.42
Yes, because the mean after taking the medicine drops from 151.6 mmHg to 142 mmHg and the standard deviation is smaller.

9. (a) Rozaidy: 25%, Wildan: 45%

(b) Wildan's performance is better than Rozaidy's because the values of the median and the third quartile are higher than Rozaidy's.

10. (a)



(b) By assuming that the car has the same percentile for 60-km and 100-km journeys, the car will consume 5.00 litres of petrol for the 100-km journey.

CHAPTER 8 Mathematical Modeling

Self Practice 8.1a

1. (a) In this problem, we know the speed of the river current and the travel time between the two places on the riverbank. We need to find the speed of the boat on calm water.

(b) Assumptions that need to be made:
- Speed of the river current and speed of the boat do not change all the time.
- The effect of friction between the surface of the boat and the river as well as the wind resistance which can affect the actual speed of the boat are ignored.

The variables involved are v for the speed of the boat, t for the travel time and s for the distance between the two places along the riverbank.

Self Practice 8.1b

1. Guide:
- The marathon runner runs 32 km per week on a regular basis and plans to increase his running distance for 10% each week. Derive a mathematical model for his running distance, $D(x)$.
- Assume the marathon runner increases his running distance for exactly 10% each week and the runner did not suffer from running related injuries.
- The variables involved are D for the running distance in km and x for the number of weeks of training.
- Construct a table as follows:

Week	Calculation (km)	Running distance (km)
1	$32 + 32 \times 0.1$ $= 32\ (1 + 0.1)$	$32(1.1)$
2	$32(1.1) + 32(1.1) \times 0.1$ $= 32(1.1)(1 + 0.1)$	$32(1.1)^2$
3	$32(1.1)^2 + 32(1.1)^2 \times 0.1$ $= 32(1.1)^2(1 + 0.1)$	$32(1.1)^3$
4	$32(1.1)^3 + 32(1.1)^3 \times 0.1$ $= 32(1.1)^3(1 + 0.1)$	$32(1.1)^4$

- The mathematical model is $D(x) = 32(1.1)^x$. This mathematical model is an exponential function that allows the runner to calculate his running distance after x weeks of training.
- The mathematical model is used for calculating the running distance with 10% increment each week. If the runner wishes to start with s km of run on a regular basis and increases his distance for r% for the subsequent week, then the mathematical model can be altered and becomes $D(x) = s\left(1 + \frac{r}{100}\right)^x$.

2. Guide:
- Why young children are more at risk of having hyperthermia compared to adults in hot weather?
- The rate of fluid loss from body depends on the surface area of a person's body.
- The amount of fluid in body depends on the volume of the body.
- The main factor is the ratio of total surface area per volume. The larger ratio of total surface area per volume will cause the body to lose fluid more quickly.
- Cubes are used to illustrate the body.
- The variables involved are y for ratio of total surface area per volume and x for length of side of cube.
- Consider cubes of different dimensions.

Length of side of cube (unit)	Total surface area (unit^2)	Volume (unit^3)	$\frac{\text{Total surface area}}{\text{Volume}}$
1	6	1	6
2	24	8	3
3	54	27	2
4	96	64	1.5
5	150	125	1.2
6	216	216	1
7	294	343	0.86

- The mathematical model is $y = \frac{6}{x}$ where y = ratio of total surface area per volume and x = length of side of cube. This mathematical model is a reciprocal function.
- Smaller cubes have larger ratio of total surface area per volume than larger cubes. Smaller cubes have a greater potential to lose fluid. Smaller cubes will lose fluid at a greater rate than larger cubes.
- Applying this to the problem of children suggests that children will lose fluid more quickly than adults. Therefore, they are at greater risk of dehydration that leads to hyperthermia.
- We can extend the approach of the ratio of total surface area per volume for cubes to the construction of physical models of children and adults, using cubes, cuboids, cylinders, cones or spheres.

Extensive Practice

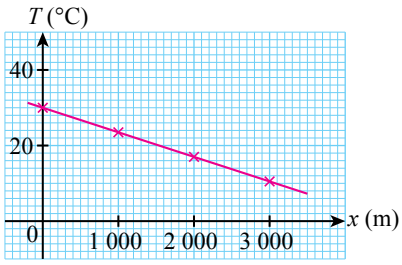
1. Identifying and defining the problems, making assumptions and identifying the variables, applying mathematics to solve problems, verifying and interpreting solutions in the context of the problem, refining the mathematical model, reporting the findings

2. (a) A – quadratic model
B – exponential model
C – linear model

(b) Exponential model. Temperature will not rise as shown in the quadratic model. Temperature will not drop below 0°C as shown in the linear model.

3. (a) $T = -(6.5 \times 10^{-3})x + 30$

(c)



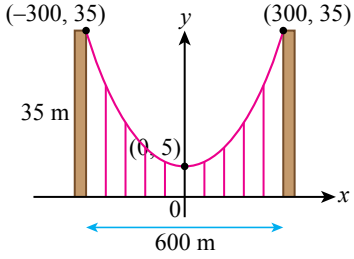
(d) From the graph, when x = 4 000, T = 4. Range of function $T(x)$ is $4°C \leq T(x) \leq 30°C$. The height of Mount Kinabalu is about 4 000 m.

(e) The gradient = -6.5×10^{-3} °C per m. It means the temperature decreases by 0.0065°C for every 1 metre up.

(f) y-intercept = 30°C. It means the temperature at the sea level (altitude 0 m) is 30°C.

(g) Yes. Since the relation between T and x is a linear function, the temperature decreases at a linear rate as altitude increases in the troposphere.

4. (a) $y = \frac{1}{3\ 000}x^2 + 5$



(b) 58

(c)

Distance between the suspender cables (m)	Type of material	Total cost of the suspender cables (RM)
15	A	848 250
20	B	831 400
25	C	781 920

(Guide: use spreadsheet to calculate length of each suspender cable required.)
Material C with the distance of 25 m between the suspender cables should be used.

co-insurance in medical insurance (*ko-insurans dalam insurans perubatan*) The policyholder bears a portion of the medical cost covered by the policy at an agreed rate.

co-insurance in property insurance (*ko-insurans dalam insurans harta*) The policyholder insures the property at a certain amount based on the percentage determined by the insurance company from the insurable value of the property.

combined variation (*ubahan bergabung*) Variation that involves a combination of direct variation or joint variation, and inverse variation.

congruency (*kekongruenan*) Objects with the same shapes and sizes.

constant (*pemalar*) A fixed or unchanged quantity value.

cumulative frequency (*kekerapan longgokan*) The sum of frequency of a class interval and all the frequencies before it in a set of data.

deductible (*deduktibel*) A part of the losses that should be borne by a policyholder before being eligible to make a claim from an insurance company.

direct variation (*ubahan langsung*) Variation that relates two variables, x and y, such that y increases (decreases) when x increases (decreases) and $\frac{y}{x}$ is a constant.

element (*unsur*) Every number in a matrix.

enlargement (*pembesaran*) A transformation in which all the points of an object move from a fixed point with a constant ratio.

frequency polygon (*poligon kekerapan*) A graph that connects the midpoints of the class intervals which lie at the upper end of each rectangle in a histogram.

general insurance (*insurans am*) An insurance policy that protects a policyholder apart from loss or damage covered by life insurance.

histogram (*histogram*) A graph that represents the frequency of each class interval with a rectangle of the same width and the height represents the frequency.

identity matrix (*matriks identiti*) A square matrix that when multiplied by any other matrix will produce the matrix itself.

income tax (*cukai pendapatan*) Tax imposed on income earned by a salaried individual or a company.

inverse matrix (*matriks songsang*) Matrix A is the inverse matrix of matrix B if the product of both matrices is an identity matrix, such that $AB = BA = I$.

inverse variation (*ubahan songsang*) Variation that relates two variables, x and y, such that y increases (decreases) when x decreases (increases) and xy is a constant.

joint variation (*ubahan tercantum*) Direct variation that relates three or more variables in which one variable varies as a product of two or more variables.

life insurance (*insurans hayat*) An agreement that guarantees payment of a monetary benefit to a policyholder due to death or under the conditions specified in the contract.

mathematical modeling (*pemodelan matematik*) A mathematical approach to understand and make decision about daily life phenomenon.

matrix (*matriks*) A set of numbers arranged in rows and columns to form a rectangular or a square array.

ogive (*ogif*) A cumulative frequency graph.

order of a matrix (*peringkat matriks*) Matrix with order $m \times n$ is a matrix that has m rows and n columns.

property assessment tax (*cukai pintu*) Tax levied on all holdings or properties in order to cover municipal expenses.

quit rent (*cukai tanah*) Tax levied on land owner.

risk (*risiko*) The possibility of an unavoidable disaster.

road tax (*cukai jalan*) Tax levied on a road user who owns a vehicle.

sales and service tax (*cukai jualan dan perkhidmatan*) Sales tax is the tax levied only once on various taxable goods at the stage of manufacturing or during importation while service tax is the tax levied on customer who receives taxable services.

scale factor (*faktor skala*) The ratio of the length of side of the image to the length of side of the object.

similarity (*keserupaan*) A pair of shapes which have proportional sides and angles of equal size.

tax (*cukai*) Payment imposed by government on individuals or companies for a country's development.

tax rebate (*rebat cukai*) Amount given to reduce total income tax to be paid.

tessellation (*teselasi*) A pattern of recurring shapes that fills a plane without leaving empty spaces or overlapping.

unit circle (*bulatan unit*) A circle with a radius of 1 unit.

variation (*ubahan*) A relation between two or more variables.

zero matrix (*matriks sifar*) A matrix with all its elements equal to zero.

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ISBN 978-967-2930-12-9
9 789672 930129
FT085002